Appendix of the laboratory guide of the experiment "Variable Friction Force Virtual Experiment"

I. Signs in the Equation of Motion

In the figures below, the reference frame *x*O*y* was chosen as defined in the laboratory guide Part I, hence the velocity of the coin is positive in the *x* direction, while in the *y* direction it begins positive, then decreases and becomes negative, with increasingly absolute value after reaching the point of maximum height ($v_y = 0$). **Figures 1** and **2** below show the force diagram from lateral and upper views respectively. Because the coin moves over the surface of the inclined plane, there is an equilibrium of forces in the direction perpendicular to the plane, that is, $N = mg\cos\theta$ all along the motion.

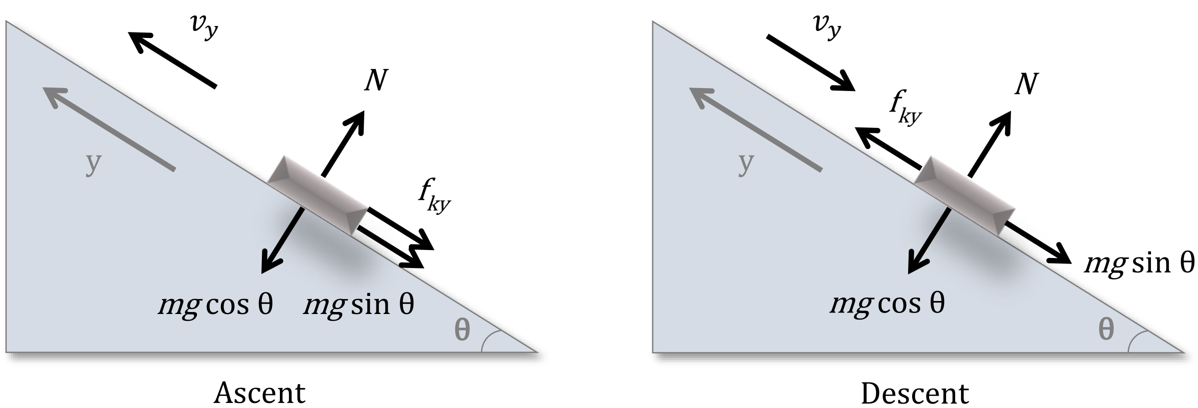

Figure 1. Diagram of the forces on the coin, where the symbols *mg*, *N* and f_{ky} represent the weight, normal force and the component *y* of the friction force, respectively, and θ, the angle between the inclined and the horizontal planes. Lateral view.

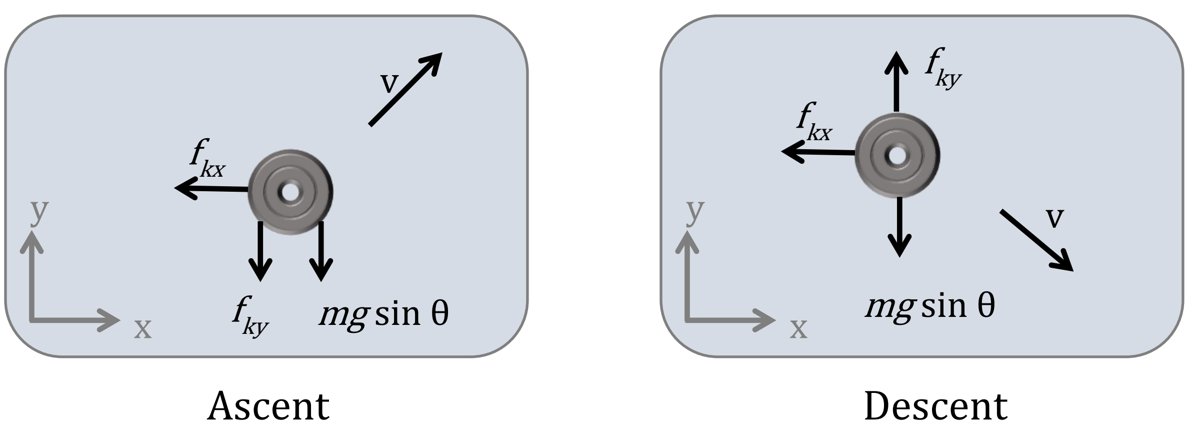

Figure 2. Upper view of the same diagram of fig. 1.
f_{kx} stands for the horizontal component of the friction force.

It is good practice to adopt the general rule of imposing an adequate sign to each datum or value (positions, velocities, gravity, friction force) to match the chosen reference frame. In this way, the deduced velocity and acceleration will be computed with the adequate signs because of the algebraic properties of the equations. Projecting the vector equation of motion, $\vec{F}_R = m \cdot \vec{a}$, on the *x* and *y* directions, gives

$$\begin{cases} F_x = m \cdot a_x \\ F_y = m \cdot a_y \end{cases}$$

With the orientation of the axis y shown on **Fig. 2**, it is seen that the coin weight (a datum) points at a direction contrary to the chosen axis. Therefore

$$F_y = f_{ky} - mg \sin\theta$$

representing the intensity of the gravitational acceleration by a constant $g > 0$. In relation to the sign of the friction force, the empirical laws of the friction applied to a body in motion are used: the force intensity is equal to $\mu_k N$, where μ_k is the kinetic friction coefficient, and has the direction opposite to the velocity vector. Therefore, each component of the friction force is opposite to the correspondent component of the velocity, whose signs are deduced from the observed positions along time. Then, from **Figs. 1** and **2**, the components of the friction force are:

$$f_{kx} = F_x = m \cdot a_x$$

$$f_{ky} = F_y + mg \sin\theta = m \cdot a_y + mg \sin\theta$$

Compare to equations (10) and (11) in Part I of the guide.

These equations transform the deduced accelerations from the measured positions into forces, which are necessary to model the mechanics of this motion, therefore the embodied signs are fundamental. The reference frame must be considered to get these signs right.

II. Friction force properties from the data

Items **B16** and **B17** of the guide to Part II propose to verify if the experimental data obtained from the recorded motion give support to the empirical contact friction laws, that define the intensity and direction of the friction force. If the experimental data do not contradict these laws, it is possible to elaborate a theoretical model to predict the coin trajectory along the plane surface. In the following, we will adopt the notation of the Part II of the guide, where equation (21) summarizes the model, which is a system of equations of motion that allow to determine the coin trajectory from the system parameters and the initial conditions of the coin launching.

Behavior of the intensity of the friction force

The empirical friction laws establish that the force intensity, f_k, is constant and proportional to the intensity of the normal force, N,

$$f_k = \mu_k N \tag{1}$$

where the proportionality constant μ_k is the kinetic friction coefficient. From the condition of equilibrium of forces in the direction normal to the plane, it is deduced that the normal force is a constant in the experimental arrangement, with intensity

$$N = mg \cos\theta \tag{2}$$

From the components of the friction force in each instant, its intensity can be obtained with:

$$f_k = \sqrt{{f_{kx}}^2 + {f_{ky}}^2} \tag{3}$$

Substituting eqs. (2) and (3) in eq. (1):

$$\sqrt{{f_{kx}}^2 + {f_{ky}}^2} = \mu_k \, mg \cos\theta \tag{4}$$

See that equations (3) and (4) are equations (12) and (13), respectively, of the guide.

Figure 3 shows the experimental values of the friction force intensity in function of the time of one of the set if images. The standard deviation of the distribution is $\sigma = 279$ g·cm/s² and the mean value and its standard deviation are $\overline{f_k} = 826(54)$ g·cm/s².

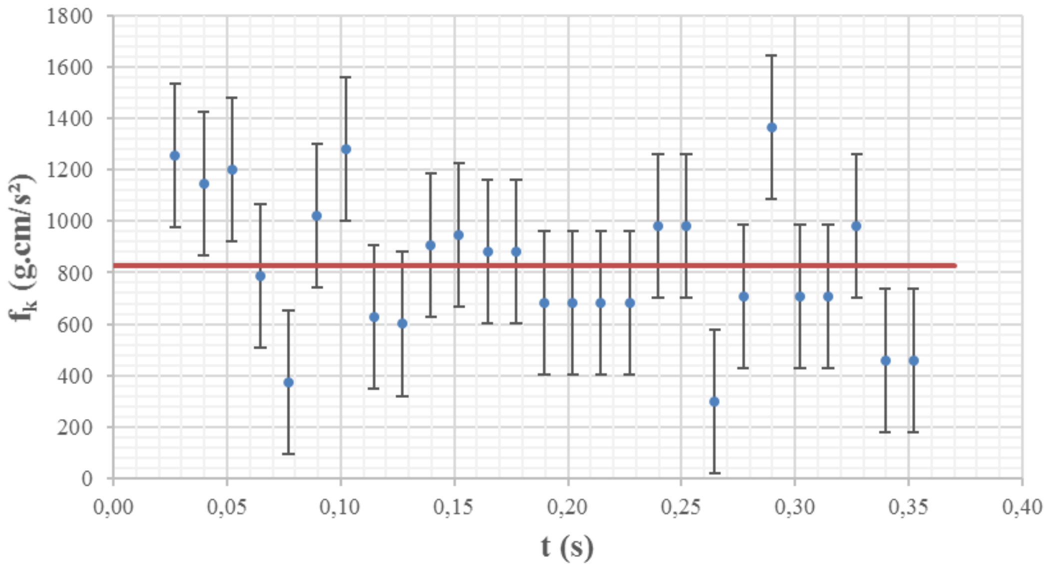

Figure 3. Experimental values of the friction force intensity in function of the time, obtained with equation (3). The uncertainty bars are associated with one standard deviation and the continuous line was drawn on the mean value of the experimental data.

When a magnitude has a constant value and the obtained data have a distribution obeying a Gaussian probability function, about 68% of the measured values fall within one standard deviation of the mean value; about 95%, within two standard deviations; probably none within more than three standard deviations, unless you take a lot of data (of the order of one hundred or more). This is exactly what happens observing the graph of **Figure 3** –18 of 27 points are closer to the mean than one standard deviation. Therefore it is plausible to adopt the intensity of the friction force as a constant and use the mean value of this distribution to determine the friction coefficient from eq. (4), $\mu_c = 0{,}29(2)$.

Evaluation of the angle between friction force and velocity

The empirical laws establish that the friction force direction is opposite to the velocity, hence the angle between these vectors must be equal to π rad in any instant. One way to validate this property is to check the compatibility of the experimental values of this angle with the expected value. Unfortunately, there is no simple way to perform this analysis. Below, we explain one of the possible methods, which simplifies the statistical interpretation of the result.

The known expressions of the vector product are merged in this calculation:

$$\vec{v} \times \vec{f}_k = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_x & v_y & 0 \\ f_{kx} & f_{ky} & 0 \end{vmatrix} = (v_x f_{ky} - v_y f_{kx})\hat{k} \tag{5}$$

and

$$|\vec{v} \times \vec{f}_k| = v \cdot f_k \cdot \sin\alpha', \qquad \text{with } \alpha' \in [0; \pi] \tag{6}$$

In equation (5), the $\hat{k}$ component can be positive, null or negative, while (6) defines α', the *smallest* angle between the vectors, always in the range $[0; \pi]$, since $\sin\alpha' \geq 0$ because all the other quantities in this equation are positive or null. The angle α found, when attaching to vector $\vec{v}$ the reference for its measurement, as illustrated by Fig. 4, can be obtained from

$$v_x f_{ky} - v_y f_{kx} = v \cdot f_k \cdot \sin\alpha, \qquad \text{with } \alpha \in [0; 2\pi] \tag{7}$$

where α is the angle measured from the velocity vector to the friction force vector in the anti-clockwise direction – check that this formula is correct. Therefore, from (7):

$$\alpha = \arcsin\left(\frac{v_x f_{ky} - v_y f_{kx}}{v \cdot f_k}\right), \qquad \text{with } \alpha \in [0; 2\pi] \tag{8}$$

It should be noted that each value of the sine in the interval $[-1; 1]$ corresponds to two arcs in the interval $[0; 2\pi]$, because the sine is positive in the first and second quadrants, and negative in the other two. For this reason, it is necessary to choose, in the domain of the sine function, which branch has to be used. In another words, it is necessary to stablish if the arcs are in the first and fourth quadrants, or in the second and third. Notice that we are not speaking about the quadrants in the *x*O*y* reference frame, but one that has the abscissa in the direction of the velocity, represented with dashed lines on Fig. 4.

If the vectors velocity and friction force are opposite, the angle α will be compatible to π rad, which is the single angle belonging to the second and the third quadrants. However, in view of the statistical fluctuation of the experimental data, the measured angle will likely be different of this value, either greater or smaller than π rad. Hence, $\arcsin\alpha \in \left[\frac{\pi}{2}; \frac{3\pi}{2}\right]$ in this problem. **Figure 4** shows, for this example, the possible configurations that may arise from these fluctuations – be careful to distinguish the quadrants in the reference frame linked to the velocity from the *x*O*y* reference frame.

The function $\arcsin(\,)$ in the electronic spreadsheet, however, returns arcs in the interval $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$, therefore the implementation of equation (8) to give the arc-sine function for the particular branch $\left[\frac{\pi}{2}; \frac{3\pi}{2}\right]$ of the function domain requires some care.

Figure 5 shows a plot of the sine function. As each invertible branch of the sine domain has a length of π rad, it is necessary to analyse how the angles in $\left[\frac{\pi}{2}; \frac{3\pi}{2}\right]$ have to be obtained from the values into $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. A first intuitive trial could be to add π to the result in $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. However, notice that moving the plot of the branch where $x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$ by π to the right, you do not obtain the plot of the sin function for $x \in \left[\frac{\pi}{2}; \frac{3\pi}{2}\right]$; these branches are mirrored, therefore:

$$\alpha = \pi - \arcsin\left(\frac{v_x f_{ky} - v_y f_{kx}}{v \cdot f_k}\right). \tag{9}$$

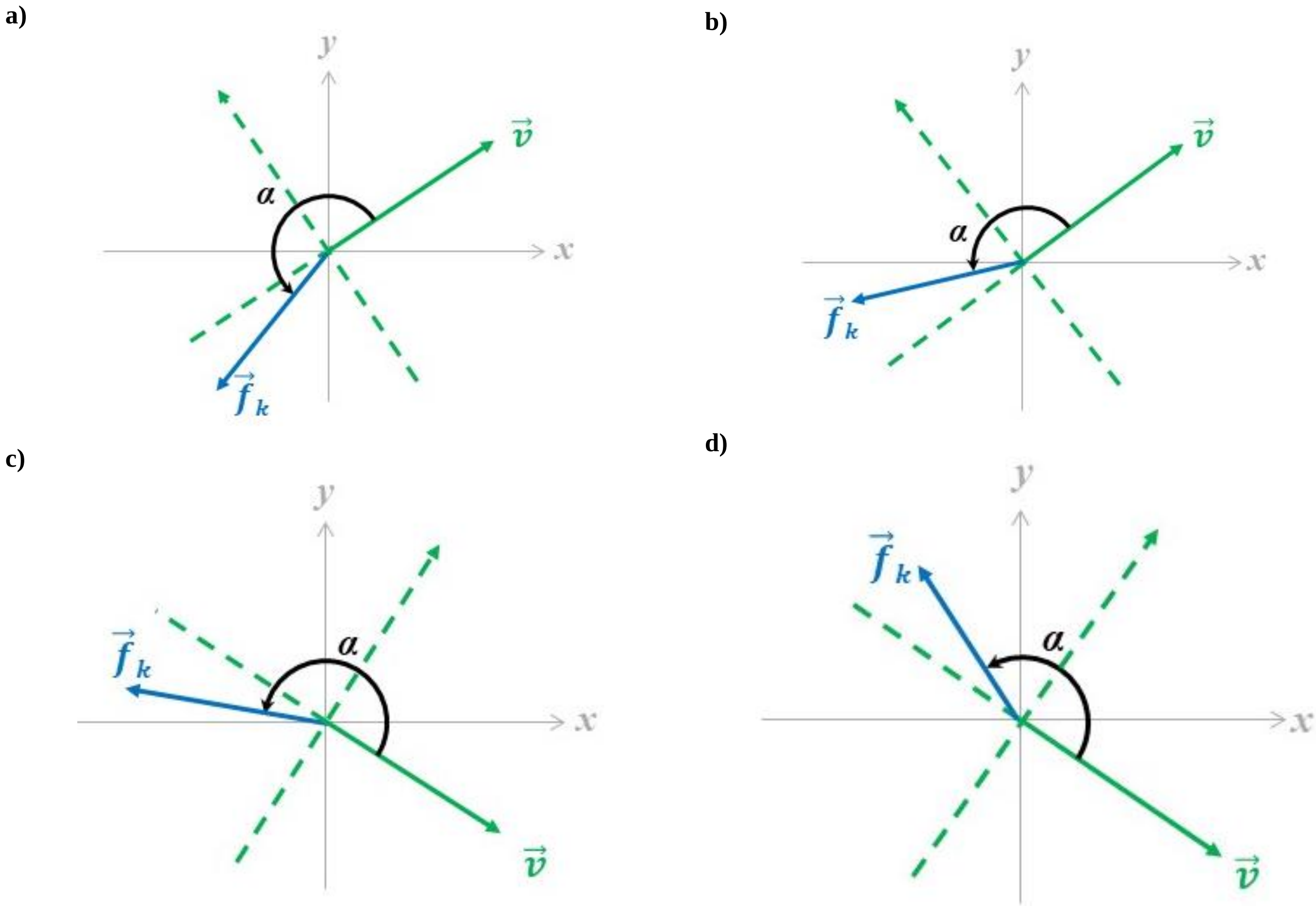

Figure 4. Possible orientations of the vectors velocity and friction force over the plane of motion.
In **a)** and **b)**, the coin has an ascending motion (velocity vector in the 1º quadrant),
while in **c)** and **d)** it is descending (the velocity vector is in the 4º quadrant).
In **a)** and **c)**, $\alpha > \pi$ and the argument of the arc-sine function in equation (8) is negative.
In **b)** and **d**), $\alpha < \pi$ and the argument of the arc-sine function in equation (8) is positive.

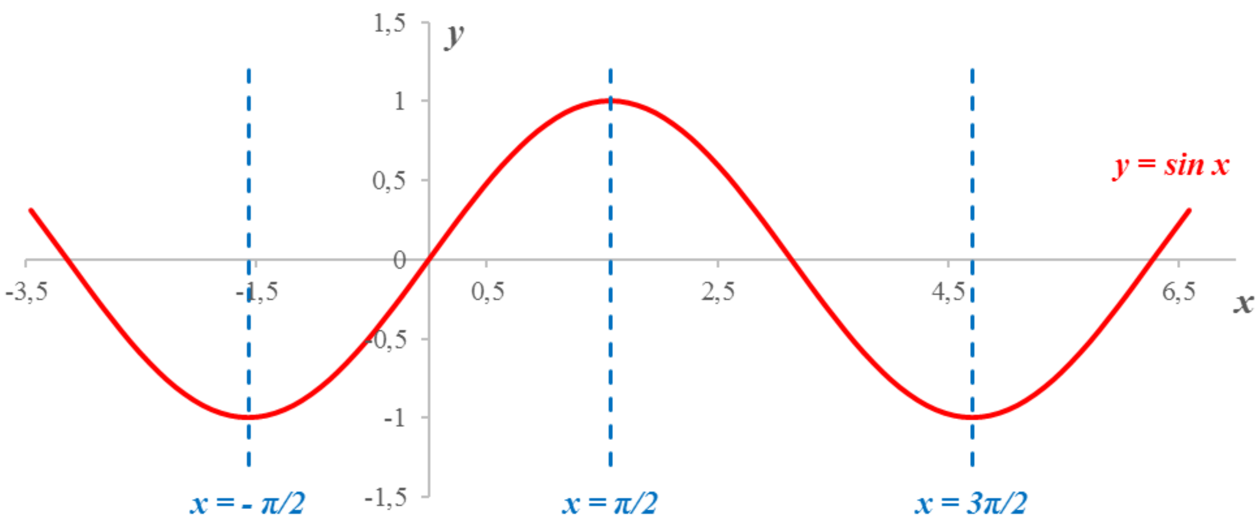

Figure 5. Plot of the function $y = \sin x$ near $x = {}^{\pi}/_{2}$.

Written in the spreadsheet syntax, formula (8) reads:

```
=PI()-ASIN((vx*fy-vy*fx)/(v*f))
```

where `vx`, `vy`, `v`, `fx`, `fy` and `f` refer, respectively, to the components *x* and *y* of the velocity (`v`) and friction force (`f`) intensities.

Evaluation of the uncertainty in the angle between friction force and velocity

The statistical fluctuation in the measurement of the angle α comes from the uncertainties in the measured components of the velocity and the friction force, which appear in eq. (9) above in a particular combination, which we define as

$$u = \frac{v_x f_{ky} - v_y f_{kx}}{v \cdot f}$$

In this experiment, the relative standard deviation in the velocity can be neglected when compared to that in the friction force, therefore a good approximation can be obtained using

$$\sigma_u^2 \cong \left(\frac{\partial u}{f_{kx}}\right)^2 \sigma_{fkx}^2 + \left(\frac{\partial u}{f_{ky}}\right)^2 \sigma_{fky}^2$$

Adopting $\sigma_{fkx}^2 = \sigma_{fky}^2 = \sigma_f^2$, after many algebraic steps, we arrive to:

$$\sigma_u \cong \frac{\sigma_f}{f}. \qquad (10)$$

Note that at the right member appears a relative standard deviation, while at the left, it is just the standard deviation (which is OK – f is dimensional, while u is not). Now, going back to formula (9), and observing the data, that give $\alpha \approx \pi$, we deduce that $u \approx 0$. Using the approximation

$$\alpha = \sin u \approx u$$

valid for $u \approx 0$, we found that the standard deviation in the angle is given by

$$\sigma_\alpha \simeq \frac{\sigma_f}{f}. \qquad (11)$$

Considering that the intensity of the friction force is constant (**Fig.** 3 shows that this can be true, within experimental uncertainties), the mean value of the friction force can be used in equation (11). This leads to the conclusion that the uncertainty in a measurement of the angle α is the same for all time instants. Finally, notice that formula (11) is valid also for the mean value $\bar{\alpha}$ when the standard deviation of the **mean** of the friction force is used, $\sigma_{\bar{f}} = 54$ g·cm/s².

Conclusion on the angle between friction force and velocity

Figure 6 shows the experimental values of α in function of the time, for the same set of images analysed in **Fig. 3**, where the uncertainty was evaluated by formula (11), using $\sigma_f = 279$ g·cm/s². On **Fig. 6**, it can be observed that 19 of 27 points are at less than one standard deviation of the expected value of π rad, and the mean value of the distribution is $\overline{\alpha} = 3.20(7)$ rad – note that 0.07 is the standard deviation of the mean value $\overline{\alpha}$, and not of each α_i data, for an specific instant t_i . Therefore, either by the statistical interpretation of the graph or by the analysis of the mean value, the experimental results are in agreement with the empirical friction law that states that the friction force is opposite to the velocity. Hence, $\alpha = \pi$ rad will be adopted. Once defined the intensity and direction of the friction force, the equation of motion of the coin can be written as shown in Part II of the guide and allows to predict the trajectory of the coin on the inclined plane surface.

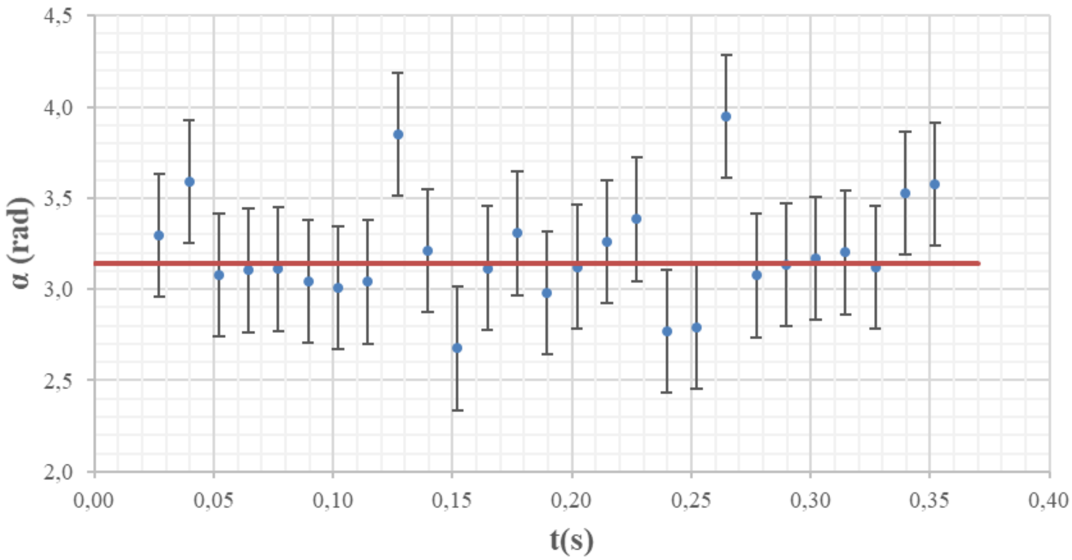

Figure 6: Experimental values of the angle between the coin velocity and the friction force in function of the time, obtained with eq. (9). The uncertainty bars correspond to one standard deviation, from eq. (11). The continuous line was drawn at the expected value of π rad.