Laboratory guide of the "Variable Friction Force Virtual Experiment" – Part II

A) Introduction

In the first part of the experiment, we determined the physical quantities related to the motion of the coin, for instance, the friction force in the observed points of the trajectory. It must now be clear that the trajectory is not a parabola, because the resultant force on the coin varies.

In this second part of the work, it will be elaborated an equation of motion, found its solution and explained the observed trajectory. We start searching for the behaviour of the magnitude and direction of the frictional force along the motion, and checking whether it conforms to the empirical friction force laws. The obtained results will lead to a theoretical model for the equation of motion that allows calculating the *x* and *y* position-time functions. Finally, the agreement between the experimental and calculated trajectories will reveal whether the model is suitable.

B) Analysis procedure (Continuation)

Here, we will use the projections of the frictional force and the velocity in the directions O*x* and O*y* of the coordinate system, determined in Part I, as well as the equations shown there.

B16. ***Behavior of the friction force magnitude.*** The empirical laws of friction applied to the coin in motion determine the vector characteristics of the frictional force: its direction is opposite to that of the velocity, and its magnitude is given by:

$$f_k = \left|\vec{f}_k\right| = \mu_k N \quad \Rightarrow \quad \sqrt{{f_{kx}}^2 + {f_{ky}}^2} = \mu_k m g \cos\theta \tag{13}$$

where μ_k is the kinetic friction coefficient and $N = \left|\vec{N}\right| = \left|\vec{W}\right| \cos\theta$, the normal force magnitude. A more involved calculation will be needed to find the orientation of the friction force relative to the velocity, described in **B17**.

Retrieve the graph of the friction force magnitude as a function of time, plotted in Part I, and draw a line parallel to the time axis at the force mean value. Verify if the spreading of points around the mean value is consistent with a friction force of constant intensity, in agreement with Equation (13); the *Appendix* of this guide suggests a methodology to perform this comparison. Once confirmed the mean value of f_k**,** use it and the known values of m, g and θ to determine the kinetic friction coefficient μ_k.

B17. ***Behavior of the friction force direction***. The components of the coin velocity and friction force on the coin have been calculated in Part I. **Figure 1** shows one of the images, where the vectors velocity and friction force were represented, and the sketch beside this image shows the angle between them. To compare the relative orientation between velocity and friction force all along the trajectory, it is necessary to measure the angle between the vectors using a uniform criterion, like that shown in the figure below.

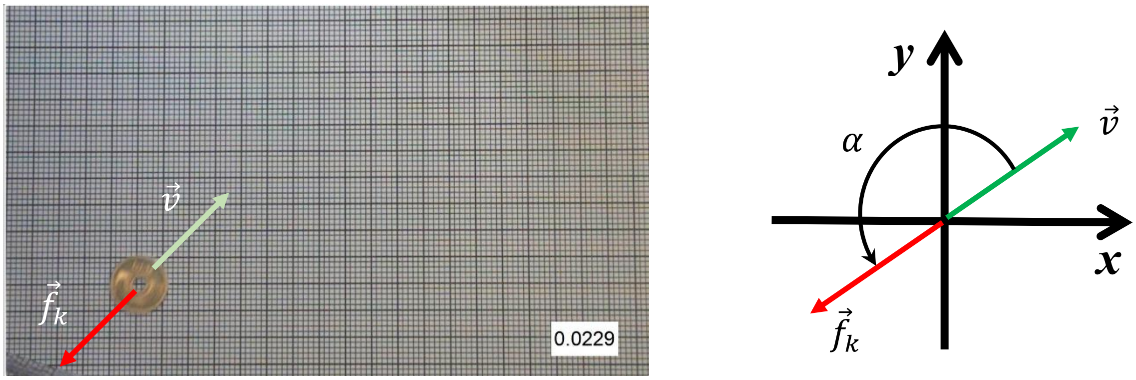

Figure 1. At left, an image extracted from the video, where the representations of the velocity and the expected friction force vectors were drawn. The sketch at right indicates the measurement of the angle in the counter clockwise direction using the velocity as the origin of the angular coordinate, for two different possible experimental results ($\vec{v}$ and $\vec{f}_k$), affected by random variations.

First, we note that the *dot* product $\overrightarrow{f_k} \cdot \vec{v} < 0$ for all instants, meaning that $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$, and the relative orientation is like that shown on the left part of Fig. 1. In this angular range, the angle α between the velocity and friction force vectors as sketched in the right part of Fig. 1 can be calculated for all the instants of time resorting to the *vector* product between the velocity and the friction force by[1]:

$$\alpha = \arcsin\left(\frac{v_x f_{ky} \quad - v_y f_{kx}}{v\ f_k}\right) \tag{14}$$

where v_x, v_y and f_{kx}, f_{ky} are the projections of the velocity and friction force in the directions Ox and Oy, respectively, and

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2} \tag{6}$$

as defined in **Part I** of this guide and will be used all along this document. Since sin θ takes the same value for many angles θ, the sine function can be inverted only under some restriction, explained in the *Appendix* of this guide, which details how to work with the arcsin function usually provided in the computer systems.

Build the graph of the angle α in function of the time and include a line parallel to the axis of time in the expected value $\alpha = \pi$ rad. Verify that the points in the graph scatter around π rad, in agreement with the expected behavior of a kinetic friction force.

B18. ***Building a model.*** Returning to eqs. (7) and (9), we remark that the resultants in the Ox and Oy directions are, respectively, the horizontal component of the friction force and the sum of the components of the friction and weight forces:

$$\vec{F}_R = f_{kx}\hat{\imath} + \left(f_{ky} + P_y\right)\hat{\jmath} \tag{15}$$

An expression that accounts for the direction and intensity of the friction force, from Eq. (13), is

[1] In formula (14), we use the *signed* projection of $\vec{v} \times \vec{f}_k$ on an axis perpendicular to the plane of motion and relate it to the angle α shown in Fig. 1. Calculating the angle between the vectors, which is in the range $[0, \pi]$, would make more difficult to check whether the velocity and friction force are opposite.

$$\vec{f}_k = \mu_k N\left(-\frac{\vec{v}}{v}\right) = \mu_k mg\cos\theta\left(-\frac{\vec{v}}{v}\right) \tag{16}$$

The term between parentheses is the versor on the velocity direction, and the minus sign means that friction acts in the opposite direction, a condition checked in **B17**.

Using eq. (16) in (15),

$$\vec{F}_R = \left(-\frac{\mu_k mg\cos\theta\, v_x}{v}\right)\hat{\imath} + \left(-\frac{\mu_k mg\cos\theta\, v_y}{v} - mg\sin\theta\right)\hat{\jmath} \tag{17}$$

Applying the 2nd Newton's Law

$$\vec{F}_R = m\vec{a} = m(a_x\hat{\imath} + a_y\hat{\jmath}) \tag{18}$$

Finally, the equations of motion are:

$$\begin{cases} a_x = \dfrac{dv_x}{dt} = -\dfrac{\mu_k g\cos\theta\, v_x}{v} \\ a_y = \dfrac{dv_y}{dt} = -\dfrac{\mu_k g\cos\theta\, v_y}{v} - g\sin\theta \end{cases} \tag{19}$$

The coin velocity in each instant of time is the solution of these coupled nonlinear differential equations, which can only be obtained numerically.

B19. *Calculating the position along time.* The time interval between successive images is sufficiently small to assume that the velocity is approximately constant and equal to the velocity at the beginning of the interval, but sufficiently large to make the velocity during the following interval different; this last velocity can be computed from the previous using the acceleration given by equation (19). In the menu "*Guias Auxiliares*" on the site, there is a guide on *Numerical Integration*, that provides an application example with more details.

If Δt is the interval between successive images, then

$$t_{n+1} = t_n + \Delta t \tag{20}$$

From the positions and velocities in $t = t_n$, the velocity at t_{n+1} is calculated using a_x and a_y from (19):

$$\begin{cases} v_{x(n+1)} = v_{x(n)} + \left(-\dfrac{\mu_k g\cos\theta\, v_{x(n)}}{\sqrt{{v_{x(n)}}^2 + {v_{y(n)}}^2}}\right)\Delta t \\ v_{y(n+1)} = v_{y(n)} + \left(-\dfrac{\mu_k g\cos\theta\, v_{y(n)}}{\sqrt{{v_{x(n)}}^2 + {v_{y(n)}}^2}} - g\,\mathrm{sen}\,\theta\right)\Delta t \end{cases} \tag{21}$$

The *x* and *y* positions at t_{n+1} can be evaluated as:

$$\begin{cases} x_{n+1} = x_n + v_{x(n)}\Delta t \\ y_{n+1} = y_n + v_{y(n)}\Delta t \end{cases} \tag{22}$$

Notice that in (21) we expanded v in its components, since these formulas are required in the calculations.

Start the process with the first values of the velocity and position obtained in the spreadsheet. Next, these new velocities and positions computed with equations (21) and (22) must be introduced at the *right* member of these equations to compute the velocity and position of the subsequent instant; this *iterative* process must be successively repeated until the last instant of interest.

B20. ***Fitting the model parameters***. Compare the computed positions calculated by numerical integration with those measured in the images. To facilitate the comparison process, draw on the same graph the experimental data (already plotted in item **B4**, Part I) and the computed values (**B18**). If the trajectories are not close, change slightly (within one or two standard deviations) *one* of the parameters μ_k, x_0, y_0, v_{0x} or v_{0y} and recalculate the worksheet (changing the value in the parameter cell and pressing *Enter* should work if you implemented adequately the equations in the spreadsheet). Begin searching for better values of v_{0x} or v_{0y}, afterwards try to adjust μ_k and change x_0 or y_0 only when you understood the effect of changes in the other parameters. Usually, after some attempts that will take few minutes, you will find a set of parameter values that give a trajectory consistent with the experimental, although likely the fit will not be perfect.

C) Procedure to elaborate the report

Write the report towards an audience that knows neither the experiment nor the analysis procedure, but who has knowledge in physics. Describe what was done, state the conclusion, and explain how it was reached. Try to be clear, objective and brief, and use your own words. Each group must hand in a single report, with the following sections:

C6. *Identification*: list the names of the group members (or write your name, if you worked alone) and the code corresponding to the case analysed.

C7. *Introduction*: explain the objectives of the experiment and how it was searched.

C8. *Experimental Description*: with your own words, make a brief description of the experimental arrangement, mention the components and their characteristics.

C9. *Obtained Results*: show your numerical results from items **B16** and **B17**, in the form of tables and graphs. Verify the units of the quantities and check whether they are represented with the suitable number of decimal places. Check that you plotted also the uncertainty bars.

C10. *Data Analysis.* Summarise why the graphs of the previous item show the validity (or not) of the empirical laws of friction and how the statistical interpretation of the uncertainty bars contributed to the conclusion. Plot, in a single graph, the measured and calculated trajectory of the coin with the measured values of the parameters μ_k, x_0, y_0, v_{0x} or v_{0y}, along with their values. Repeat this graph, but now using the trajectory calculated with the fitted the parameters (according to item **B20**), and show the best values for μ_k, x_0, y_0, v_{0x} and v_{0y}. **Do not include** the raw data nor those extracted from the initial analysis of the set of images because they have been included in the brief report of Part I.

C11. *Discussion.* Based on the tests done in items **B16** and **B17**, justify why you considered valid the empirical laws of friction or show how you have found that they were not valid. Show possible discrepancies between the measured and calculated trajectories, from item **C10**, and suggest potential causes of discrepancies. Comment if the fitted values of μ_k, x_0, y_0, v_{0x} and v_{0y} are in agreement with the measured values and, if you found discrepancies, explain the likely reasons. Describe how the trajectory of the coin would change if the following parameters were varied:

a. angle of the plane inclination;
b. angle of the coin launching;
c. friction coefficient between the coin and the contact surface;
d. intensity of the initial coin velocity;
e. coin mass;
f. gravitational acceleration.

C12. *Conclusion.* Return to the introduction, focus on the objective and comment if you consider that it was fully, partially, or not attained. *Suggestion*: return to items **C5** (*Part I*) and **C11** (*Part II*), and comment how the empirical friction laws were validated and how the application of a model to predict the trajectory enables to deepen the study of the mechanics of a body subjected to a variable force.