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Nonlinear vortex-phonon interactions in a Bose–Einstein condensate

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Abstract

We consider the nonlinear coupling between an exact vortex solution in a Bose–Einstein condensate and a spectrum of elementary excitations in the medium. These excitations, or Bogoliubov–de Gennes modes, are indeed a special kind of phonons. We treat the spectrum of elementary excitations in the medium as a gas of quantum particles, sometimes also called *bogolons*. An exact kinetic equation for the bogolon gas is derived, and an approximate form of this equation, valid in the quasi-classical limit, is obtained. We study the energy transfer between the vortex and the bogolon gas, and establish conditions for vortex instability and damping.

Keywords: Bose–Einstein, condensates, vortices, turbulence, phonons, bogolons

(Some figures may appear in colour only in the online journal)

1. Introduction

The area of Bose–Einstein condensates (BECs), specially those produced with laser cooled low density alkaline gases, has been explored in the last two decades in many different directions [1, 2]. BECs are indeed quantum fluids, which like the classical fluids allow for the existence of sound waves, usually called elementary excitations or Bogoliubov modes. The formation of quantum vortices can be considered as one of their most remarkable properties.

Vortices in BECs are nonlinear structures which have been studied by many authors, in both experiments [3, 4] and theory [1, 5]. Similarities with Rossby waves, as those existing in the rotating atmosphere of planets, has also be explored [6]. Abrikosov arrays or lattices of quantum vortices can be excited, and display oscillations called Tkachenko modes [7, 8], as first observed by [9]. Rossby–Tkachenko modes, corresponding to a general class of lattice oscillations, can also be considered [10].

In this work, we consider the interaction of vortices with a spectrum of elementary excitations in the medium. These excitations, or more generally Bogoliubov–de Gennes (BdG) modes, are indeed a special kind of phonons. BdG modes can be seen as generalizations of Bogoliubov modes, valid for a non-uniform medium. Here we assume that the condensate is uniform, if we ignore the influence of the confining potential. Otherwise, it will be in a Thomas–Fermi equilibrium and will satisfy the usual parabolic density profile. However, the existence of a turbulent spectrum of fluctuations will modify this equilibrium, in such a way that the condense will appear, for each BdG mode inside the spectrum, as a non-uniform or a modified Thomas–Fermi equilibrium [11]. The vortex solutions will then be defined on this modified equilibrium. We treat the spectrum of elementary excitations in the medium as a gas of quantum particles, sometimes also called *bogolons*. This could be relevant to the excitation of vortices in a turbulent BEC [12]. In the present context, bogolons are used to identify bosonic excitations in the condensate, but they can also be used in a more general context, to design fermionic excitations in superfluids and superconductors [13].

Starting from the usual BdG mode equations, we derive an equivalent wave-kinetic equation describing the evolution of an appropriate Wigner function. Wigner functions were used in the past to describe the condensate itself [10]. But here the Wigner functions are used to describe the BdG or bogolon modes, while both the background condensate and

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the vortices are described with the usual mean field wave functions.

The vortex dispersion relation in the presence of an arbitrary bogolon spectrum is derived. Conditions for the excitation and damping of vortices due to the presence of a BdG or bogolon spectrum are established. Special cases are considered explicitly. In a generic turbulent medium, several vortices co-exist with the bogolon spectrum. Here we focus on the elementary processes associated with a single vortex, and although the interaction energy between two vortices could also be modified by the background spectrum, this will not be addressed in the present work.

The structure of the paper is the following. In section 2, we present the basic equations describing the BEC in the mean field approximation. In section 3, using the the auto-correlation function for two distinct pairs of time and positions, the kinetic equation for the bogolon gas is derived. Section 4 focuses on the evolution of a single vortex in a turbulent background described by the appropriate Wigner quasi-probability function. In section 5 the question of vortex energy increase or damping is addressed in terms of the new kinetic theory. Finally in section 6 our conclusions are collected.

2. Basic formulation

We describe the evolution of BEC using the mean-field approximation. For that purpose, we start with the GP equation, which can be written as

$$i\hbar\frac{\partial}{\partial t}\psi = (H_0 + g\,|\psi|^2)\psi, \quad H_0 = -\frac{\hbar^2\nabla^2}{2m} + U_0(\mathbf{r}). \quad (1)$$

Here we use the standard notation, where ψ is the condensate order parameter, $U_0(\mathbf{r})$ the confining potential, and g the coupling constant. Let us assume a generic solution of the form

$$\psi(\mathbf{r}, t) = [\psi_0(\mathbf{r}, t) + \tilde{\psi}(\mathbf{r}, t)]\exp(-i\mu t/\hbar), \quad (2)$$

where ψ_0 describes the slow condensate field, and $\tilde{\psi}$ is a high frequency perturbation, which can be associated with turbulence. It will be used to describe a superposition of elementary excitations in the medium. Here, μ is the chemical potential. Replacing this in equation (1), and averaging over a time interval much longer than the period of the turbulent fluctuations, to be specified later, we get for the average condensate field

$$i\hbar\frac{\partial\psi_0}{\partial t} = [H_0 + g(n_0 + 2n_T) - \mu]\psi_0, \quad (3)$$

where we have used the two density variables, $n_0 = |\psi_0|^2$ characterizing the average or equilibrium condensate density, and $n_T = \langle|\tilde{\psi}|^2\rangle$ obtained by averaging over the turbulent fluctuations. Here, we should notice that $\langle\tilde{\psi}\rangle = 0$ and $\psi_0 \equiv \langle\psi\rangle$. Subtracting equation (3) from (1), we obtain for the fast component of the matter field

$$i\hbar\frac{\partial\tilde{\psi}}{\partial t} = (H_r + g|\tilde{\psi}|^2)\tilde{\psi} + 2g(|\tilde{\psi}|^2 - n_T)\psi_0 + g(\psi_0^2\tilde{\psi}^* + \psi_0^*\tilde{\psi}^2). \quad (4)$$

with the new Hamiltonian $H_r = H_0 + 2gn_0(\mathbf{r}) - \mu$. We should notice that we have, in this equation, $|\tilde{\psi}|^2 = n_T + \delta|\tilde{\psi}|^2$, where n_T is the slow part and $\delta|\tilde{\psi}|^2$ contains the high frequency mixing of the BdG spectrum. Such spectrum can be explicitly described as

$$\tilde{\psi}(\mathbf{r}, t) = \sum_k [u_k(\mathbf{r})\exp(-i\omega_k t) + v_k^*(\mathbf{r})\exp(i\omega_k t)], \quad (5)$$

where each mode is identified by the quantity k, representing a set of discrete numbers of a continuum of wavevectors, ω_k are the eigenfrequencies, and the pair of functions $u_k(\mathbf{r})$ and $v_k(\mathbf{r})$ are the corresponding BdG field components. Replacing this in equation (4), and neglecting the nonlinear mode mixing terms, we get the usual BdG equations for each mode

$$(\hbar\omega_k - H_r)u_k = g\psi_0^2 v_k, \quad (\hbar\omega_k + H_r)v_k = -(g\psi_0^2)^* u_k. \quad (6)$$

In homogeneous condensates, and in a broad range of situations discussed in our previous work [11], we can assume solutions that satisfy the equations $\nabla^2(u_k, v_k) = -k^2(u_k, v_k)$. We can then easily solve equations (6) and derive the mode dispersion relation

$$\hbar\omega_k = \sqrt{(H_k + 2gn_0 - \mu)^2 - (gn_0)^2}, \quad (7)$$

with $H_k = \hbar^2k^2/2\,m + U_0(\mathbf{r})$. If we associate the turbulence fluctuations to short wavelengths, much shorter than the size of the condensate, we can neglect the confining potential and set $U_0 = 0$, as well as $\mu = gn_0$. We are then reduced to the well known expression

$$\omega_k = \sqrt{c_s^2k^2 + \frac{\hbar^2k^4}{4m^2}}, \quad c_s = \sqrt{\frac{gn_0}{m}} \quad (8)$$

Here c_s is the Bogoliubov sound speed. A generalization of this dispersion relation to twisted BdG modes in homogeneous, cylindrical and toroidal geometries can be found in [11]. At this point we introduce $u_k = |u_k|\exp(i\varphi_u)$ and $v_k = |v_k|\exp(i\varphi_v)$. The mode energy, or density, can then be written as

$$n_k = |\psi_k|^2 = |u_k|^2 + |v_k|^2 + 2|u_k||v_k|\cos(\varphi_u - \varphi_v) \quad (9)$$

For mode components in quadrature, we have $\cos(\varphi_u - \varphi_v) = 0$, and the usual Bogolioubov norm implies that $|u_k|^2 = 1 + |v_k|^2$. Other, zero-energy modes could also be considered [14], for which $|u_k|^2 = |v_k|^2$. At this point, it should be noticed that equations (6) can be reduced to

$$[\hbar^2\omega_k^2 - (H_r^2 - |g\psi_0^2|^2)]u_k = 0. \quad (10)$$

It is useful to introduce the new quantities

$$U_k(\mathbf{r}, t) = u_k(\mathbf{r})\exp(-i\omega_k t), \quad V_k(\mathbf{r}, t) = v_k(\mathbf{r})\exp(-i\omega_k t) \quad (11)$$

They obviously satisfy the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \omega_k^2\right)(U_k, V_k) = 0, \tag{12}$$

where ω_k is determined by equation (8). This will be useful in the study of the nonlinear coupling between the BdG modes (or *bogolons*) and a vortex, as shown next.

3. Kinetic equation for the bogolon gas

In order to study the energy transfer between a slowly varying perturbation, more specifically, a single vortex, and a background spectrum of turbulent fluctuations, we use for the slow component of the condensate wavefunction

$$\psi_0(\mathbf{r}, t) = \psi_{00}(\mathbf{r}) + \psi_v(\mathbf{r}, t), \tag{13}$$

where the first term is the steady-state part of the condensate, and ψ_v is the disturbance associated with the vortex. We now have

$$|\psi_0|^2 = n_0 + \psi_{00}\psi_v^* + \psi_{00}^*\psi_v, \quad n_0 = |\psi_{00}|^2 + |\psi_v|^2. \tag{14}$$

We notice that, for $U_0 = 0$ and $\mu = gn_0$, we have

$$H_r^2 - |g\psi_0^2|^2 = -\frac{\hbar^2\nabla^2}{2m}\left(-\frac{\hbar^2\nabla^2}{2m} + gn_0\right) - g^2[2n_0(\psi_{00}\psi_v^* + \psi_{00}^*\psi_v) + (\psi_{00}\psi_v^* + \psi_{00}^*\psi_v)^2]. \tag{15}$$

This expression can be obtained using equation (14) and the definition of H_r, stated after equation (4). Replacing this in equation (10), using (11), and neglecting the second order contributions from the vortex perturbation, we are then led to the conclusion that, in a condensate perturbed by a vortex ψ_v, the BdG or bogolon mode can be described by

$$\left[\frac{\partial^2}{\partial t^2} - \left(gn_0 - \frac{\hbar^2\nabla^2}{2m}\right)\frac{\nabla^2}{2m} - G\right](U_k, V_k) = 0, \tag{16}$$

with the auxiliary function $G \equiv G(\mathbf{r}, t)$ defined by

$$G = 2\frac{g^2 n_0}{\hbar^2}(\psi_{00}\psi_v^* + \psi_{00}^*\psi_v), \tag{17}$$

where the quadratic terms in ψ_v were ignored. It can easily be seen that this mode equation reduces to equation (12), above, when the vortex disappears and $\psi_v = 0$. This perturbed mode equation has now to be coupled to the evolution equation for the vortex field ψ_v, which in turn will depend on the mode functions (U_k, V_k), as shown below.

But, before considering the vortex equation, it is useful to replace equation (16) by a wave-kinetic equation capable of describing an arbitrary superposition of BdG modes, or in other words, an arbitrary bogolon gas. For that purpose, we follow the standard Wigner–Moyal procedure [10], focusing on the field U_k, given the symmetry with V_k. We start by introducing the auto-correlation function for two distinct pairs of time and positions, as defined by

$$K_{12} = U_k(\mathbf{r}_1, t_1)U_k^*(\mathbf{r}_2, t_2) \equiv U_1 U_2^*. \tag{18}$$

From equation (16) we obtain the evolution equation for this quantity as

$$\left[\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2} + c_s^2(\nabla_1^2 - \nabla_2^2) - \frac{\hbar^2}{4m^2}(\nabla_1^4 - \nabla_2^4) + G_1 - G_2\right]K_{12} = 0, \tag{19}$$

where we have used the obvious notation $G_j \equiv G(\mathbf{r}_j, t_j)$, for $(j = 1, 2)$. We now define new pairs of space variables $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, and $\mathbf{s} = \mathbf{r}_2 - \mathbf{r}_2$, and similarly for time $t = (t_1 + t_2)/2$, and $\tau = t_2 - t_1$, and introduce the double Fourier transformation

$$K_{12} \equiv K(\mathbf{r}, t, \mathbf{s}, \tau) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\int \frac{\mathrm{d}\omega}{2\pi} W(\mathbf{r}, t, \mathbf{k}, \omega)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{s} - \mathrm{i}\omega\tau). \tag{20}$$

The new function $W \equiv W(\mathbf{r}, t, \mathbf{k}, \omega)$ is the Wigner function for the BdG field. Replacing this in equation (19) we are then able to derive the following equation determining the evolution of W

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_k\cdot\nabla\right)W = \frac{1}{\omega}G\sin(\Lambda W). \tag{21}$$

This is the wave-kinetic equation for the bogolon field, as described by the Wigner function W. Here, we have used the bogolon group velocity $\mathbf{v}_k$, as determined from the above dispersion relation

$$\mathbf{v}_k = \frac{\partial\omega_k}{\partial\mathbf{k}} = \left(c_s^2 + \frac{\hbar^2 k^2}{2m^2}\right)\frac{\mathbf{k}}{\omega}. \tag{22}$$

In equation (21) we have also used the double-sided operator

$$\Lambda = \frac{1}{2}\left(\overleftarrow{\nabla}\cdot\frac{\overrightarrow{\partial}}{\partial\mathbf{k}} - \frac{\overleftarrow{\partial}}{\partial t}\frac{\overrightarrow{\partial}}{\partial\omega}\right), \tag{23}$$

where $\overleftarrow{\nabla}$ and $\overleftarrow{\partial}/\partial t$ act backwards on G, whereas $\overrightarrow{\partial}/\partial\mathbf{k}$ and $\overrightarrow{\partial}/\partial\omega$ act forward on W. The wave-kinetic equation (21) can also be written in another equivalent form, as

$$\mathrm{i}\left(\frac{\partial}{\partial t} + \mathbf{v}_k\cdot\nabla\right)W = \frac{1}{2\omega}\int\frac{\mathrm{d}\mathbf{q}}{(2\pi)^3}\int\frac{\mathrm{d}\Omega}{2\pi}G(\Omega, \mathbf{q}) \times [W^- - W^+]\exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r} - \mathrm{i}\omega t). \tag{24}$$

In this new equation we have used the quantities $W^\pm \equiv W(\omega \pm \Omega/2, \mathbf{k} \pm \mathbf{q}/2)$, and the Fourier components

$$G(\Omega, \mathbf{q}) = \int \mathrm{d}\mathbf{r}\int \mathrm{d}t\; G(\mathbf{r}, t)\exp(-\mathrm{i}\mathbf{q}\cdot\mathbf{r} + \mathrm{i}\omega t) \tag{25}$$

In the wave-kinetic description of the BdG mode field we can also assume that the frequency ω_k of each mode $\mathbf{k}$ is determined by its linear dispersion relation. This assumption is sometimes called the *particle approximation*, and justifies the use of a reduced Wigner function, defined as

$$W(\mathbf{r}, t, \mathbf{k}) = 2\pi W(\mathbf{r}, t, \mathbf{k}, \omega)\delta(\omega - \omega_k) \tag{26}$$

This reduced form of W will be used in the following.

To complete our discussion, let us assume the quasi-classical or geometric optics approximation, where the bogolons can be described as classical quasi-particles. In this limit, diffraction and other phase effects are neglected, and we can use $\sin\Lambda \simeq \Lambda$ in equation (21). This quasi-classical approximation can also be recovered from equation (24), by using the development

$$W^{\pm} \simeq W \pm \frac{\Omega}{2}\frac{\partial W}{\partial \omega} \pm \frac{\mathbf{k}}{2} \cdot \frac{\partial W}{\partial \mathbf{k}} \tag{27}$$

Replacing this in (24), we obtain a Vlasov-type of equation, with the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_k \cdot \nabla + \mathbf{F}_k \cdot \frac{\partial}{\partial \mathbf{k}}\right) W = 0. \tag{28}$$

Here, the quantity $\mathbf{F}_k = -\nabla(G/2\omega_k)$ plays the role of a force acting on the BdG quasi-particles.

4. Vortex in a bogolon field

We consider now the evolution of a vortex in a turbulent background, as described by the Wigner quasi-distribution $W(\mathbf{r}, t, \mathbf{k})$. For simplicity, we use the geometric optics approximation. Generalization to the exact wave-kinetic description involves an heavier description, but is straightforward. The turbulent gas of bogolons is then described by equation (28), while the vortex is described by equation (3). At this point, it should be noticed that

$$n_T \equiv \langle|\tilde{\psi}|^2\rangle = \int W(\mathbf{r}, t, \mathbf{k}) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}. \tag{29}$$

Using equation (13), this allows us to rewrite equation (3) in the new form

$$\mathrm{i}\hbar\frac{\partial \psi_v}{\partial t} = [H_0 + gn_0 - \mu]\psi_v + 2g\psi_{00}\int W(\mathbf{r}, t, \mathbf{k})\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}. \tag{30}$$

We can also assume that the static mean field ψ_{00} is determined by the condition, $[H_0 + gn_0 - \mu]\psi_{00} = 0$. This will determine the Thomas–Fermi density profile. We have also neglected the ψ_v contribution to the last term of equation (30), which is valid for the perturbative analysis to be discussed here. In the nonlinear saturation regime, this contribution would have to be included.

At this point, we assume a generic vortex solution of the form $\psi_v(\mathbf{r}, t) = \Psi(\mathbf{r})\exp(-\mathrm{i}\Omega t)$. We can also write, for the bogolon gas distribution, $W(\mathbf{r}, t, \mathbf{k}) = W_0(\mathbf{r}) + \delta W(\mathbf{r}, t, \mathbf{k})$, where $\delta W(\mathbf{r}, t, \mathbf{k}) = W_v(\mathbf{r}, \mathbf{k})\exp(-\mathrm{i}\Omega t)$ is the perturbation of the bogolon gas, induced by the presence of the vortex. Replacing this in equation (30), we get

$$\hbar\Omega\Psi = [H_0 + gn_0 - \mu]\Psi + 2g\psi_{00}\int W_v(\mathbf{r}, \mathbf{k})\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}. \tag{31}$$

In order to derive a closed equation for the quantity Ψ, we need to relate W_v to Ψ, which can be done with the wave-kinetic equation (28). Noting that

$$\mathbf{F}_k = -\frac{g}{2\omega_k}\psi_{00}^*\nabla\Psi, \tag{32}$$

we obtain

$$(-\mathrm{i}\Omega + \mathbf{v}_k \cdot \nabla)W_v = \frac{g}{2\omega_k}\psi_{00}^*\nabla\Psi \cdot \frac{\partial W_0}{\partial \mathbf{k}}. \tag{33}$$

To proceed further, we take the plausible assumption that the spatial structure of W_v has the same shape of the vortex itself, which allows us to write $W_v(\mathbf{r}, \mathbf{k}) = \Psi(\mathbf{r})A_v(\mathbf{r}, \mathbf{k})$. The last term in equation (31) becomes equal to

$$2g\psi_{00}\int W_v(\mathbf{r}, \mathbf{k})\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} = -g^2|\psi_{00}|^2\Psi\mathbf{Q} \cdot \int \frac{(\partial W_0/\partial \mathbf{k})}{\omega_k(\mathrm{i}\Omega - \mathbf{v}_k \cdot \mathbf{Q})}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}. \tag{34}$$

Here, we have introduced the new vector function $\mathbf{Q} \equiv \mathbf{Q}(\mathbf{r})$, such that $\mathbf{Q} = \nabla\Psi/\Psi$. Similarly, equation (33) can be written in terms of the quantity A_v, as

$$A_v(\mathbf{r}, \mathbf{k}) = -\frac{g}{2\omega_k}\psi_{00}^*\mathbf{Q} \cdot \frac{(\partial W_0/\partial \mathbf{k})}{(\mathrm{i}\Omega - \mathbf{v}_k \cdot \mathbf{Q})}. \tag{35}$$

It is useful to notice that, in the absence of bogolon turbulence, the vortex solution would imply that $\Omega = \mu/\hbar$. The vortex solution would therefore be determined by the simple equation

$$[H_0 + gn_0 - 2\mu]\Psi(\mathbf{r}) = 0. \tag{36}$$

For a vortex around the z-axis, the corresponding solution would therefore be of the form $\Psi(\mathbf{r}) = R(r)\Phi(z)\exp(\mathrm{i}l\theta)$, where the integer l is the vortex charge, and cylindrical coordinates were used. The presence of turbulence introduces an energy correction to the vortex, ϵ, as determined by

$$\hbar\Omega = \mu + \epsilon \tag{37}$$

Assuming that equation (36) is still satisfied, we can reduce equation (31) to

$$\epsilon\Psi = 2g\psi_{00}\int W_v(\mathbf{r}, \mathbf{k})\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}. \tag{38}$$

Finally, using equation (30), and integrating over the entire volume V of the condensate, we obtain

$$\epsilon = -\frac{g^2}{V}\int_V \mathrm{d}\mathbf{r}\, n_{00}(\mathbf{r})\mathbf{Q} \cdot \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\frac{(\partial W_0/\partial \mathbf{k})}{\omega_k(\mathrm{i}\Omega - \mathbf{v}_k \cdot \mathbf{Q})}. \tag{39}$$

This is the main result of the present paper. It gives the energy correction to the vortex due to the presence of an arbitrary spectrum of BdG turbulence, as described by the unperturbed Wigner function W_0.

5. Vortex stability

We can now analyze the problem of vortex stability and the possible exchange of energy between the vortex and the bogolon spectrum of elementary excitations. First, we notice that the vector function $\mathbf{Q}(\mathbf{r})$ can be written in a more explicit form as

$$\mathbf{Q} \equiv \frac{\nabla\Psi}{\Psi} = \frac{\mathbf{e}_r}{L_r} + \frac{\mathbf{e}_z}{L_z} + \mathrm{i}l\frac{\mathbf{e}_\theta}{r}, \tag{40}$$

with

$$L_r^{-1} = \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}r}, \quad L_z^{-1} = \frac{1}{\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}z}, \tag{41}$$

where $R(r)$ and $\Phi(z)$ define the unperturbed form of the vortex. As a simple example, let us consider the case where all the BdG modes propagate along the z-axis, as described by the simple Wigner function $W_0(\mathbf{k}) = (2\pi)^2 W_0(k_z)\delta(\mathbf{k}_\perp)$. This should not be confused with a laminar flow, and small deviations with respect to the z-direction can be accommodated. Equation (39) is then reduced to

$$\epsilon = -\frac{g^2}{V}\int_V \mathrm{d}\mathbf{r}\, n_{00}(\mathbf{r}) \int \frac{\mathrm{d}k_z}{2\pi} \frac{(\partial W_0/\partial k_z)}{\omega_k(\mathrm{i}\Omega L_z - v_k)}. \tag{42}$$

Let us now focus on the imaginary part of this energy correction, $\Gamma = \Im(\epsilon) = \Im(\Omega)$, which describes the possible occurrence of an instability. We get

$$\Gamma = \frac{g^2}{V}\int_V \mathrm{d}\mathbf{r}\, n_{00}(\mathbf{r}) \int \frac{\mathrm{d}k_z}{2\pi} \frac{\mu L_z}{\omega_k} \frac{(\partial W_0/\partial k_z)}{(\Gamma L_z + v_k)^2 + \mu^2 L_z^2}. \tag{43}$$

For a nearly homogeneous condensate with $n_{00}(\mathbf{r}) \simeq n_0 =$ etc, and for $\mu L_z \gg (\Gamma L_z, v_k)$, this can be approximately written as

$$\Gamma \simeq \frac{g^2 n_0}{\mu V}\int_V \frac{\mathrm{d}\mathbf{r}}{L_z} \int \frac{\mathrm{d}k_z}{2\pi} \frac{1}{\omega_k} \frac{\partial W_0}{\partial k_z}. \tag{44}$$

And, using the Bogoliubov dispersion relation, we finally get

$$\Gamma \simeq \frac{g^2 n_0}{\mu V} c_s \int\int \frac{L_z(\mathbf{r})}{\omega_k^2} W_0(\mathbf{r}, k_z) \frac{\mathrm{d}k_z}{2\pi}\mathrm{d}\mathbf{r}. \tag{45}$$

This result clearly shows that, in the condensate regions where turbulence is present, and if $L_z < 0$, we have damping of the vortex due to its interaction with the phonons. It means that the phonons tend to gain energy. In contrast, if $L_z > 0$, the vortex grows at the expense of the turbulence energy. In both cases, the vortex will become unstable, and eventually decay into other vortex solutions, with emission or absorption of bogolons. The final result of the instability cannot be described by the present linear stability analysis. Only in the case of $L_z = 0$ can we strictly say that the vortex remains stable in the presence of turbulence. This will be the case of a vortex aligned with the z-axis.

But we can also consider vortices with finite curvature, as those discussed in detail by [17, 18], and shown in figure 1. It obviously has $L_z > 0$ for $z < 0$, and $L_z < 0$ for $z > 0$. We can see from equation (45) that such a curved vortex will remain stable if immersed in homogeneous turbulence, or when the turbulence region is symmetrically located with respect to the vortex line (figure 1(a)). And it will become unstable under the action of asymmetric turbulence, as illustrated in figure 1(b). A variety of situations can therefore occur where the value and sign of Γ will depend on the configuration of the bogolon spectrum, as defined by $W_0(\mathbf{r}, k_z)$, and on the way it occupies the vortex volume.

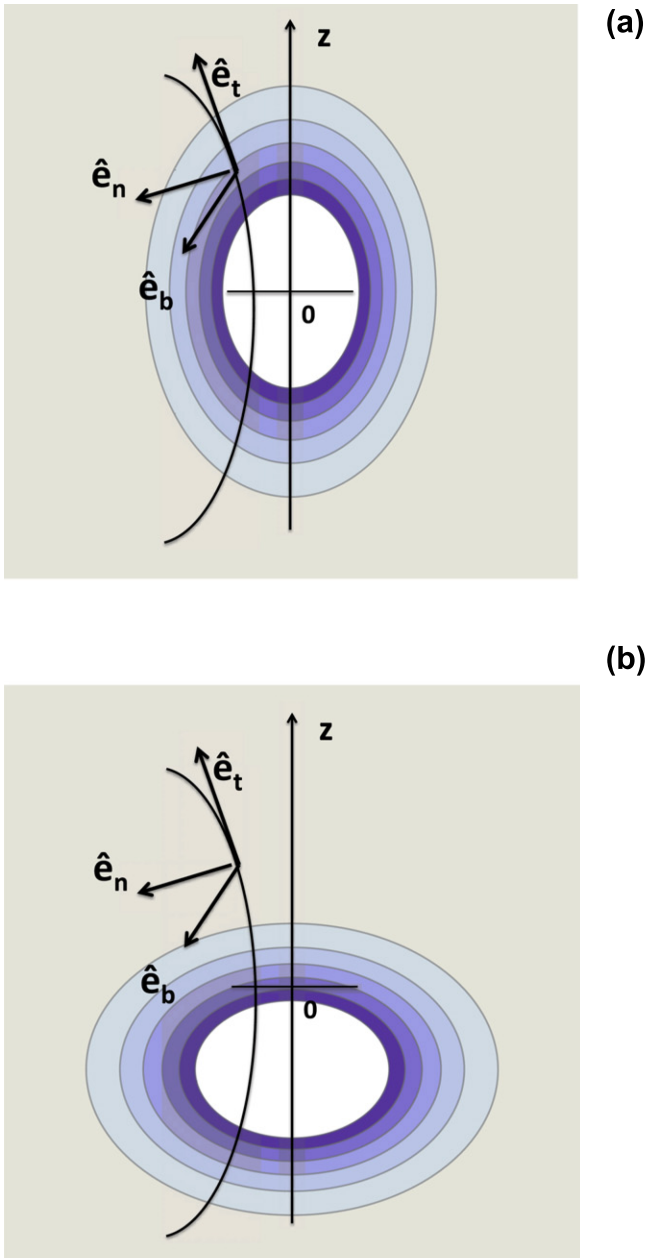

Figure 1. Vortex with a finite curvature in a bogolon gas. The vortex line is represented, as well as the condensate region occupied by turbulence, with an integrated bogolon spectrum $W_0(\mathbf{r}) = W_0(0)$ $\exp[-(z - z_0)^2/2\sigma_z^2 - r_\perp^2/2\sigma_r^2]$, (with $W_0(0) = 1$ in arbitrary units): (a) stable configuration, $z_0 = 0$ and $\sigma_z/\sigma_r = \sqrt{2}$; (b) unstable configuration, $z_0 = -1$ and $\sigma_z/\sigma_r = 1/\sqrt{2}$, when the bogolon gas covers an asymmetric part of the vortex line.

Many different configurations could lead to a non-uniform and local excitation of the turbulent field. In current experiments on condensate turbulence [12], vortices usually coexist with BdG type of oscillations. However, if we want to demonstrate the changes induced by the interactions between vortices and turbulence, we need to compare the vortex behavior with and without the oscillations. This cannot easily be done in the experiments. Therefore, the easiest way to study the evolution of a vortex in the presence of turbulence is to start with numerical simulations, using realistic condensate configurations. We can refer, for instance, to the excitation of BdG modes by the supersonic flow of a condensate past an obstacle [15]. A possible example could then be the study of vortex stability, in the vicinity of a moving obstacle (see [16] for a similar configuration). This is being considered by the authors, and will be presented in a future work.

Finally, we would like to note that the vortex itself will be forced to move due to existence of turbulence. Adapting the analysis of [18] to the present problem, we can easily conclude that the local velocity $\mathbf{v}(\mathbf{r})$ of a vortex line in the presence of an arbitrary bogolon distribution $W_0(\mathbf{r}, k_z)$ is given by

$$\mathbf{v}(\mathbf{r}) = \frac{3l\hbar}{2m\mu}\ln\left(\frac{R_\perp}{|l|\xi}\right)(\hat{\mathbf{e}}_z \times \nabla_\perp)\int W_0(\mathbf{r}, \mathbf{k})\frac{d\mathbf{k}}{(2\pi)^3}. \quad (46)$$

Here $R_\perp$ represents the condensate dimensions in the perpendicular direction, and ξ is the healing length. This expression is valid for a vortex with weak curvature, if we neglect the confining potential. The evolution of a vortex in the bogolon gas will eventually modify the present stability analysis.

It should also be noticed that the mechanism associated with damping or amplification of vortices is not explicitly described by the present perturbation method. The vortex energy could eventually change due to a small variation of the mean density in the vortex region, or a variation in curvature and length of the vortex line. For a limited and non-uniform condensate, displacement of the vortex to (or from) the boundaries would also lead to an energy variation.

6. Conclusions

We have studied the vortex-phonon interactions in a Bose–Einstein condensate. We have considered the case where a single vortex interacts with an arbitrary spectrum of elementary excitations, or BdG modes, which we have associated with a bogolon field. Starting from a generic form of BdG equations, we have derived a wave-kinetic equation which determines the evolution of the bogolon field. The field is described by a Wigner function, and can be seen as a gas of quasi-particles, the bogolons, which correspond to phonons propagating in a condensed quantum gas. Exact and approximate versions of the wave-kinetic equation where stated. In the quasi-classical approximation, the wave-kinetic equation reduces to a Vlasov-type of equation, and the Wigner quasi-distribution reduces to a classical distribution function.

Using perturbative analysis we were then able to derive the growth rate of a vortex in the turbulent field, and characterized the possible regimes where instability can eventually take place. The present analysis shows that, in general, a finite exchange of energy takes place between a vortex and the surrounding oscillations, which could be useful to future analysis of simulations and experiments. The present stability analysis is valid in the geometric optics or quasi-classical approximation, where the typical wavelength of the bologons is much smaller than the size of the vortex. But the same approach can be used for the general case, if instead of the Vlasov equation we use the exact wave-kinetic equations for the bogolon field. We have also brought attention to the occurrence of vortex motion in the presence of turbulence, which will eventually modify the vortex stability. The possible existence of stable vortex-bogolon configurations is a very interesting but difficult problem, which will be analyzed elsewhere.

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