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Bose–Einstein Condensates with Inhomogeneous Scattering Length in One and Two Dimensions¶

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Abstract—We report on investigations of the properties of bright solitons in Bose–Einstein condensates in the presence of point-like spatial inhomogeneities, in one and two dimensions. By considering an analytical variational approach and full numerical simulations, we describe such processes due to interactions between the soliton and the inhomogeneity as the trapping, reflection, and transmission of bright matter solitons. We also study the critical number of particles as a function of the magnitude of the impurity.

1. INTRODUCTION

In the last ten years in particular, Bose–Einstein condensates (BECs) in vapors of alkali-metal atoms have been under intensive experimental and theoretical investigation. The first reports of BECs in the laboratory appeared in 1995 [1]. Among the interesting forms of matter waves that can be produced in BECs are solitons, nonlinear wavepackets that keep their shape during propagation. They are formed due to the balance between the dispersion (quantum pressure) and the nonlinearity in the Gross–Pitaevskii equation (GPE). As they are robust wavepackets under perturbations, they could (due to their properties of stability) have important technological applications, from atomic interferometry to atom lasers.

Dark solitons occur when the two-body scattering length (a_s) is positive (repulsive interaction). They appear as holes in the background of the condensates [2]. Bright matter wave solitons (BS) occur in BECs when $a_s < 0$. They are harder to observe than dark solitons ($a_s > 0$) due to instabilities in two and three dimensions (2D and 3D) [3, 4]. Such condensed systems are unstable when the number of atoms N is larger than critical value N_c (e.g., 1500 to 6000 ^{7}Li atoms).

Here we report on our investigations of BS in BECs with artificially induced local inhomogeneities in one and two dimensions (1D and 2D), by changing the space distribution of the condensate. This is possible either by optical methods [5] (in which spatial variation is achieved using detuned laser fields), or by Feshbach resonance techniques (by applying an external magnetic field [6]). In our approach, we consider full numerical and variational solutions of the static and dynamic GPEs, as described in the next section. Our description of solitary atomic waves considers numerical simulations of the GPE with delta-function spatial inhomogeneity. In the case of 1D, the results are shown to be supported by full numerical calculations in 3D [7].

2. THE BOGOLIUBOV AND GROSS–PITAEVSKII FORMALISMS

In his pioneering work of 1947 [8], Bogoliubov introduced the mean field prescription of dilute Bose gases: he separated the BECs' contribution from the bosonic field operator. For dilute systems, the interaction term can be attributed only to binary collisions (low energies) characterized by a single parameter: the s-wave scattering length a_s. Following this prescription, a lowest order theory for the excitations in the interacting Bose gas is reached. If the corresponding hypothesis is valid in the limit of zero temperature (all atoms in the ground state), we need for the validity of the mean field approach $N \gg 1$ and a scattering length much smaller than the average distance between the atoms. If an external harmonic trapping potential is included in the mean field, we obtain the well-known Gross–Pitaevskii equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r}, t) = \Big[-\frac{\hbar^2}{2m}\nabla^2 + \frac{m}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2) + \frac{4\pi\hbar^2 a_s}{m}|\Psi(\vec{r}, t)|^2\Big]\Psi(\vec{r}, t), \quad (1)$$

where $\Psi \equiv \Psi(\vec{r}, t)$ is the wave function normalized to N, m is the mass of the atom and ω_i ($i = 1, 2, 3$) are the frequencies in the three spatial dimensions. A quasi-1D regime is achieved when $\omega_\perp \equiv \omega_1 = \omega_2 \gg \omega_3$; a quasi-2D regime is the result when $\omega_\perp \equiv \omega_1 = \omega_2 \ll \omega_3$.

¶ The text was submitted by the authors in English.

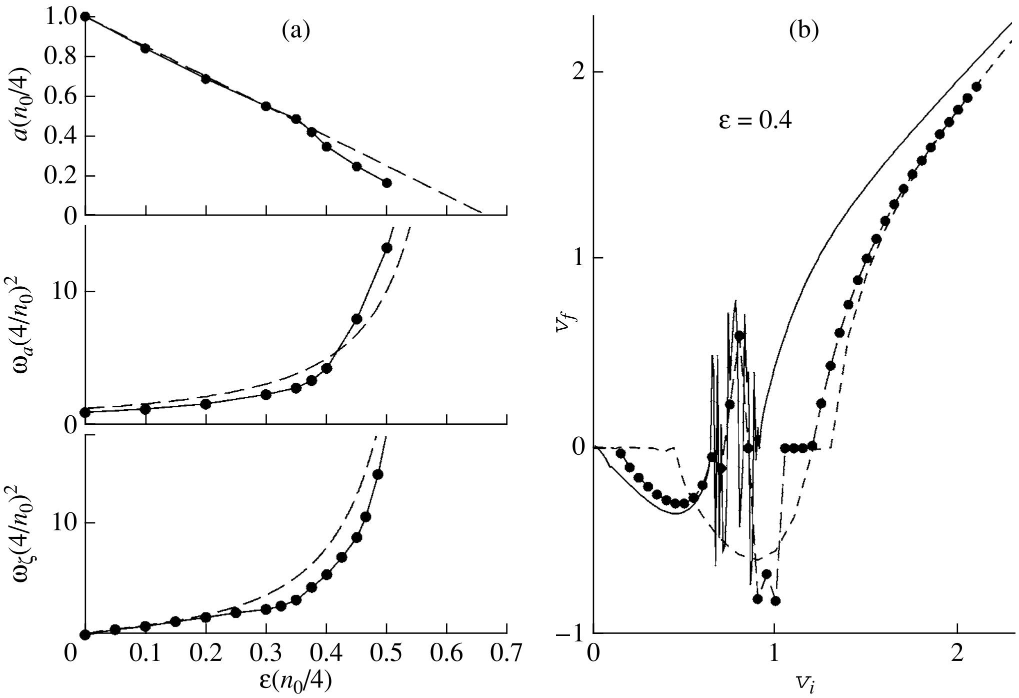

Fig. 1. (a) Numerical (solid) and variational (dotted) results are compared for the width (a), width oscillation frequencies (ω_a), and center-of-mass oscillation frequencies (ω_ζ). They are shown as functions of $\epsilon(n_0/4)$. (b) Final velocity v_f is shown (against initial velocity v_i) for $\epsilon = 0.4$ and $n_0 = 4$. Full numerical results (dashed line) are compared with variational calculations (solid and dotted–dashed lines). The dotted–dashed line is the variational results with a damping factor. All quantities in (a) and (b) are dimensionless.

2.1. A 1D Model with Delta Nonlinear Inhomogeneity

For $\alpha \equiv (\omega_3/\omega_\perp) \ll 1$, we can use the following approximation: $\Psi(\vec{r}, t) = R(x_1, x_2)Z(x_3, t)$, where $R(x_1, x_2) \equiv R$ satisfies the 2D harmonic oscillator equation. Next, we can redefine Z (which is normalized to N) along with the variables, such that $u(z, \tau) \equiv \sqrt{4|a_{s0}|}\, Z(x_3, t)e^{i\omega_\perp t}$, $\tau \equiv \omega_\perp t/2$ and $z \equiv x_3/l_\perp$, where $l_\perp \equiv \sqrt{\hbar/(m\omega_\perp)}$. Integrating over transversal directions, we obtain a 1D equation in z and τ, for $a_s < 0$. The space variation of a_s is given by a function $f(z)$ with effective strength ϵ: $a_s \equiv a_{s0}[1 + \epsilon f(z)]$. The final 1D equation is

$$i\frac{\partial u}{\partial \tau} = -\frac{\partial^2 u}{\partial z^2} + \alpha z^2 u - (1 + \epsilon f(z))|u|^2 u, \qquad (2)$$

where $f(z) = \delta(z)$. The normalization is redefined as $n_0 \equiv 4N|a_{s0}|/l_\perp$. The validity of the 1D model is expected for $n_0^2 \ll 8$ and the system will collapse at $n_0 = 4k_\perp$ ($k_\perp = 0.676$). In realistic cases, i.e., quasi-1D (cigar-like) traps, the formation and propagation of matter wave solitons has already been observed in gases of ^{7}Li atoms [4] for the trap frequencies given by $\omega_\perp = 2\pi \times 625$ Hz and $\omega_3 = 2\pi \times 3.2$ Hz, with a_s tuned to $a_{s0} = -3a_0$ (here, a_0 is the Bohr radius).

For a variational dynamic description of solitons perturbed by inhomogeneities, we consider the trial function (normalized to $n_0 = 2aA^2$)

$$u = A\,\mathrm{sech}\left(\frac{z-\zeta}{a}\right)e^{i[\phi + w(z-\zeta) + b(z-\zeta)^2]}, \qquad (3)$$

where A, a, ζ, ϕ, w, and b are time-dependent variational parameters. From the corresponding Lagrangian, we can obtain [7] the coupled equations for a and ζ. As in the Rice experiments [4], $\alpha = (\omega_3/\omega_\perp)^2 = 2.6 \times 10^{-5}$, and we use $\alpha = 0$ in Eq. (2).

A fixed point for the soliton center exists when $\zeta = 0$; the BS is trapped by a local variation of a_s. Positive variation ($\epsilon < 0$) indicates the reflection of the soliton by the inhomogeneity. The stationary width a_c is given by: $a_c = (8 - 3\epsilon n_0)/(2n_0)$.

A collapse occurs for $\epsilon \le \epsilon_c$ under the condition $d^2\langle z^2\rangle/d\tau^2 < 0$. Hence, we obtain

$$\epsilon > \frac{1}{2|u(0)|^4}\int\left(4\left|\frac{\partial u}{\partial z}\right|^2 - |u|^4\right)dz. \qquad (4)$$

The critical limit for the solitonic ansatz [7] and $a \longrightarrow 0$ corresponds to $\epsilon_c = 8/(3n_0)$.

The results of our simulations for a 1D model using variational and full numerical approaches are presented in Figs. 1–3. In Fig. 1, the variational approach is compared with the full numerical results. The delta function

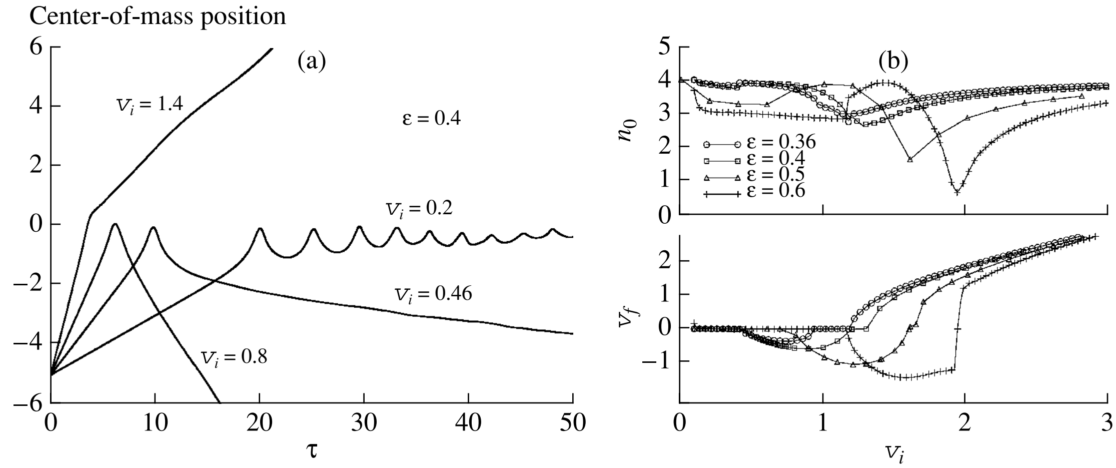

Fig. 2. (a) Numerical evolution of the center-of-mass position for $\epsilon = 0.4$, $n_0 = 4$ and different values of v_i. (b) Dimensionless numerical dependence of rescaled number of atoms n_0 (top frame), and final velocity v_f (bottom frame), with respect to the initial velocity v_i. All the quantities in (a) and (b) are in dimensionless units.

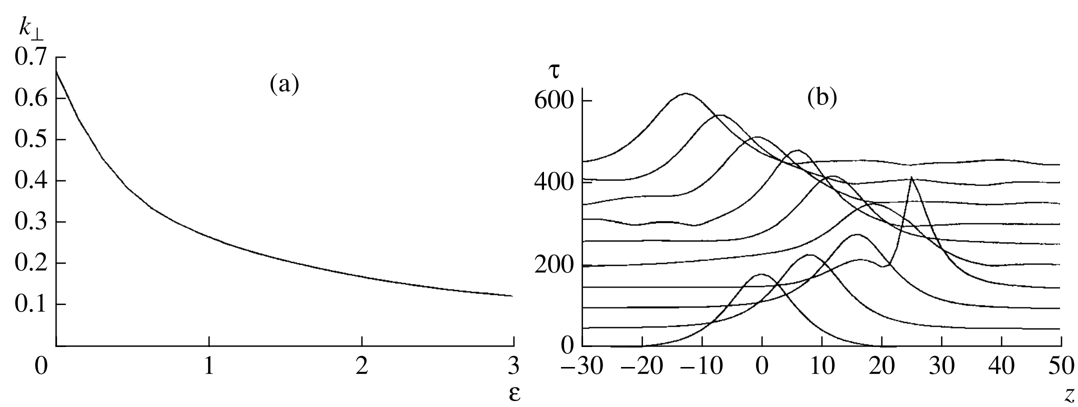

Fig. 3. (a) Critical numbers $k_\perp = N_c|a_{s0}|/l_\perp$, against the parameter ϵ, for a delta-like impurity. (b) 3D numerical evolution of a soliton reflecting at a delta-like impurity placed at $z = 25$, for $2\pi n_0 = 5$, initial velocity $v_i = 0.16$ and $\epsilon = 2$, in dimensionless units.

is simulated by a rectangular shape with base Δz and height $1/\Delta z$, where Δz is the grid step [9]. In Fig. 1a, we show the stationary results. The fixed points for the width (a), width oscillation frequencies (ω_a) and the center-of-mass oscillation frequencies (ω_ζ) are given as functions of the strength of the nonlinear delta-like impurity, ϵ. We note that the variational ansatz starts to fail near the critical point, where we must implement corrections because of the radiation. In Fig. 1b, we show the dynamic behavior of the final velocity (v_f) as function of the initial velocity (v_i), for $\epsilon = 0.4$ and $n_0 = 4$. In this case, we observe a region of initial velocity where the attractive nonlinear impurity reflects the soliton. Numerical simulations of the variational equations reproduce qualitatively the numerical results. In the numerical simulations, we obtained a window between two trapped regions with soliton reflection. In the variational simulations, the window of reflection is found at smaller initial velocities, but without a trapped region. Near the reflection points, we have more complicated dynamics. For $0.5 < v_i < 1.0$, there is noise, a behavior that occurs in systems of variational equations. One way of improving this situation is a variational approach in which the radiative friction on the soliton motion influences the impurity.

The dynamics of a BS matter-wave soliton interacting with inhomogeneity shows different regimes of propagation as we vary ϵ. In Fig. 2, we show full numerical simulations for the evolution of the center-of-mass position (for $\epsilon = 0.4$ and $n_0 = 4$) in Fig. 2a, and the results for v_f versus v_i (for different strengths of ϵ) in Fig. 2b. We assume $N|a_{s0}|/l_\perp = n_0/4 = 1$, but this can be rescaled to a value smaller than 0.676, consistent with the BECs' quasi-1D results [7]. In Figs. 2a and 2b, as in Fig. 1b, we also observe a region of velocities in which the soliton is reflected. Numerical results show that there is for each ϵ one window (interval) on the velocity axis in which there is reflection of the soliton. As shown in Fig. 2b, we observe different regimes for the soliton interaction and nonlinear impurity: reflec-

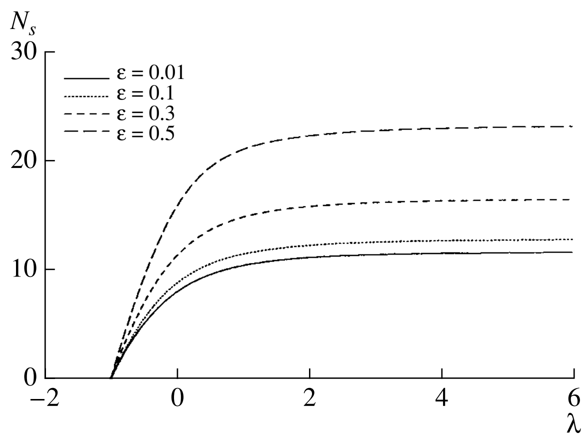

Fig. 4. Full numerical results for the number of particles N_s for a 2D condensate versus the soliton parameter λ (chemical potential) in dimensionless units. Considered here is a nonlinear step-function impurity.

tion, transmission and trapping. The reflection process occurs in the interval $0.42 < v_i < 1.2$.

3. FULL NUMERICAL 3D RESULTS

In order to check the validity of the 1D reduction when using Dirac-delta nonlinear impurity, a full numerical 3D calculation was performed in [7]. Here, we demonstrate the relation between two critical limits for the delta-like impurity in Fig. 3: $k_\perp(\epsilon) \equiv n_{0,\max}/4$ and the inhomogeneity parameter ϵ_c. For $\epsilon = 0$: $k_\perp \approx 0.676$. Plots of the soliton profile in Fig. 3b confirm that the soliton reflects at the impurity, as in the 1D case.

4. TWO-DIMENSIONAL MODEL

The case of standing bright solitons in inhomogeneous condensed media has also been studied in two dimensions, with nonlinear impurity. Here, we show some results that include a nonlinear step-function impurity (in the radial dimension) and harmonic trapping. We multiply the cubic nonlinear term in the GPE equation by $f(\rho) = [1 - \epsilon\theta(R - \rho)]$. Thus, R is the position of the impurity, and we consider solutions with the form $\phi(\rho, t) = \exp(i\lambda t)\psi(\rho)$, where λ corresponds to the soliton parameter in the dimensionless form of the corresponding stationary 2D GPE. In Fig. 4, we see the behavior of the number of particles of the standing bright solitons as a function of the magnitude of the nonlinear impurity ϵ, for $R = 1.25$. We realize that the critical limit is strongly affected by the magnitude of the impurity.

5. CONCLUSIONS

We have reported stationary and dynamic solutions of BEC with attractive interactions in the presence of a delta-like impurity in quasi-1D (and, with a step-like impurity, in 2D). In the quasi-1D regime, the bright matter wave solitons exhibit different behavior in their interaction with the nonlinear impurity: the trapping, transmission, and reflection of the soliton. For attractive nonlinear impurity, we verified the collapse of the soliton if $\epsilon > \epsilon_c$. The variational approach provides a good description of the collapse and a good qualitative description of the reflection and trapping dynamics. Our full numerical 3D calculation supports the 1D results. In the 2D case, we considered a nonlinear step impurity and found that the critical number of atoms depends strongly on the strength of the impurity.

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REFERENCES

1. M. H. Anderson *et al.*, Science **269**, 198 (1995); K. B. Davis *et al.*, Phys. Rev. Lett. **75**, 3969 (1995); C. C. Bradley *et al.*, Phys. Rev. Lett. **75**, 1687 (1995).
2. J. Denschlag *et al.*, Science **287**, 97 (2000); S. Burger *et al.*, Phys. Rev. Lett. **83**, 5198 (1999).
3. L. Khaykovich *et al.*, Science **296**, 1290 (2002).
4. K. Strecker, G. Patridge, A. Truscott, and R. Hulet, Nature **417**, 15 (2002).
5. F. K. Fatemi, K. M. Jones, and P. D. Lett, Phys. Rev. Lett. **85**, 4462 (2000).
6. E. Timmermans *et al.*, Phys. Rep. **315**, 199 (1999).
7. F. Kh. Abdullaev, A. Gammal, and L. Tomio, J. Phys. B **37**, 635 (2004).
8. N. N. Bogolyubov, J. Phys. (Moscow) **11**, 23 (1947).
9. Y. S. Kivshar, Zhang Fei, and L. Vazquez, Phys. Rev. Lett. **67**, 1177 (1991).