Physics

Teachers' guide

Supplementary mathematics

Nuffield Advanced Science

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Nuffield Advanced Science

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Foreword

It is almost a decade since the Trustees of the Nuffield Foundation decided to sponsor curriculum development programmes in science. Over the past few years a succession of materials and aids appropriate to teaching and learning over a wide variety of age and ability ranges has been published. We hope that they may have made a small contribution to the renewal of the science curriculum which is currently so evident in the schools.

The strength of the development has unquestionably lain in the most valuable part that has been played in the work by practising teachers and the guidance and help that have been received from the consultative committees to each Project.

The stage has now been reached for the publication of materials suitable for Advanced courses in the sciences. In many ways the task has been a more difficult one to accomplish. The sixth form has received more than its fair share of study in recent years and there is now an increasing acceptance that an attempt should be made to preserve breadth in studies in the 16–19 year age range. This is no easy task in a system which by virtue of its pattern of tertiary education requires standards for the sixth form which in many other countries might well be found in first year university courses.

Advanced courses are therefore at once both a difficult and an interesting venture. They have been designed to be of value to teacher and student, be they in sixth forms or other forms of education in a similar age range. Furthermore, it is expected that teachers in universities, polytechnics, and colleges of education may find some of the ideas of value in their own work.

If the Advanced Physics course meets with the success and appreciation I believe it deserves, it will be in no small measure due to a very large number of people, in the team so ably led by Jon Ogborn and Dr Paul Black, in the consultative committee, and in the schools in which trials have been held. The programme could not have been brought to a successful conclusion without their help and that of the examination boards, local authorities, the universities, and the professional associations of science teachers.

Finally, the Project materials could not have reached successful publication without the expert assistance that has been received from William Anderson and his editorial staff in the Nuffield Science Publications Unit and from the editorial and production teams of Penguin Education.

K. W. Keohane
Co-ordinator of the Nuffield Foundation Science Teaching Project

Introduction

'Furthermore, in physics, the ability to think effectively depends upon having some rather definite skills and knowledge, particularly on having some mathematical understanding.'
Nuffield Advanced Physics Teachers' handbook.

Some parts of mathematics which are especially valuable in the education of a scientist are taught within the Advanced Physics course, along with the physics for which they are used. These are mainly the solving of simple differential equations, and the use of the exponential, sine, and cosine functions. Students who also take a sixth form mathematics course should have no special difficulties with these, but others may need more time for practice than the Physics course can allow. There may be some who have difficulty with mathematics learned before the sixth form, for example, with proportion, powers of ten, logarithms, and so on.

This *Guide* is intended to assist teachers who have students who are not doing a sixth form course in mathematics, and to provide material from which they can select what they want.

The *Guide* is written in a sequence which follows the needs of the Nuffield Advanced Physics course fairly closely, as they develop. The methods suggested have also been chosen to fit closely with the particular needs of the Physics course. There is, of course, no suggestion that there are no better methods or sequences.

Each numbered sub-section of the *Guide* is followed immediately by a selection of examples relevant to that piece of work. Further examples may well be needed, a number of which can be taken from the Advanced Physics course (see table 1).

This *Guide* is deliberately fairly comprehensive. Besides containing revision of elementary mathematics, which many students will not need, it contains some parts which, for satisfaction and completeness, go rather beyond the minimum requirements of the Physics course. It is not a mathematical course to be followed all through, but a collection of resources from which to select a course. Details of books referred to in the text are given in the list on page 118.

It would be valuable if the teacher responsible for the physics can also teach any supplementary mathematics that is needed. In that way, the work done in the time allocated to the latter can be of the greatest support and value to the physics. If this cannot be arranged, then the closest collaboration will be necessary to ensure that the work in physics and mathematics is carefully co-ordinated.

To mathematicians who receive this *Guide* apologies are necessary. To suggest that they should adopt these methods in preference to others of their own would be an impertinence. No doubt ways will occur to them of developing the mathematical ideas so as to assist the Physics course. Details of the Nuffield Advanced Physics course are to be found in the *Teachers' handbook*, and a glance at that volume will be necessary for teachers of mathematics who do not know this course.

General outline

Computation

This is mainly revision of ground usually covered at O-level – indices, logarithms, slide rule work, etc. Many teachers will feel they can omit it altogether or make the treatment brief. The need for practice in these topics lies in their constant use in the Physics course.

Functions and graphs

The functions dealt with are those which students will meet in physics. Those not mathematically inclined will need lots of practice to help to establish a feeling for the functional patterns and to enable them to recognize those patterns in the future.

Differentiation and integration

The emphasis is on graphical methods throughout, with frequent reference to physical situations in which a derivative has some significance. Numerical methods play a prominent part in this work.

Exponential changes and probability

Growth and decay patterns are an important part of the Physics course, and the concept of probability appears in Unit 9, *Change and chance*. The weaker mathematicians will undoubtedly need some extra time to become familiar with the ideas involved.

Timing

The time required will depend on students' needs, but is likely to fall within a range from 2 periods a week for one year, to 3 or 4 per week over the same time. A less desirable allowance would be 2 periods a week spread over two years; it is less desirable because much of the work and extra practice will come too late to be of great help.

It is not easy to suggest how much time should be spent on the various topics – much will depend on the ability of members of the class and whether the early work can be covered quickly – but the aim should be for students to spend most of the available time doing rather than hearing about mathematics. Activities which develop confidence, such as handling numbers, drawing graphs, and using the slide rule successfully, will be the most helpful way to spend time.

Table 1 gives a rough guide to the points at which the various sections are used in the first year of the Physics course, together with an approximate time scale.

Stage in first year of the Physics course	**Sections of the Physics course**	**Mathematics**
Term 1	Unit 1, *Materials and structure*	computation
	Unit 2, *Electricity, electrons, and energy levels*	functions and graphs
		differentiation
Term 2	Unit 3, *Field and potential*	sine and cosine functions
	Unit 4, *Waves and oscillations*	integration
Term 3	Unit 5, *Atomic structure*	exponential change
	Unit 6, *Electronics and reactive circuits*	probability
		dynamics

Table 1

Section 1

Computation

This section is mainly concerned with methods of computation – indices, logarithms, the slide rule – and it also has a brief mention of areas, volumes, and trigonometrical ratios in 'Scaling'. It is, therefore, a revision of O-level work. Many teachers will no doubt omit it altogether, feeling that the ideas contained in it are well enough established already. Others taking the weaker mathematicians may judge that benefit will accrue from going 'back to the beginning' to see where it all stemmed from. In any case, teachers should not spend long on this revision – in fact, there is much to be said for practice in arithmetic at regular intervals throughout the course to build up students' speed and confidence in handling numbers.

The need for practice in these methods lies in their use in the Physics course. In the lessons devoted to mathematics, the aim should be to give students the necessary background knowledge and a sufficient number of examples, so that progress in physics is not delayed or the emphasis shifted too much from the physics to the mathematics.

The ideas in 'Scaling' (1.4) appear early in Unit 1, *Materials and structure*. Some teachers may wish to do section 1.4 before section 1.1.

1.1 Indices – rules for manipulation

Often difficulties stem from an inability to translate the 'shorthand' of mathematics into ordinary language. Students must know what the index means and should be encouraged to express things like 2^5 in words, e.g. 2^5 means that 2 is multiplied by itself 5 times, i.e. $2\times2\times2\times2\times2$ which equals 32. 2^5 is another way of writing 32.

Easy manipulation of numbers in this index form requires rules which are not difficult to establish when the indices are simple positive integers. For this, it would be wise to work with numbers throughout (e.g. $2^3\times2^2 = \overline{2\times2\times2}\times\overline{2\times2} = 2^5$), and generalize only at the end. Writing out in full should establish that:

a $p^m\times p^n = p^{m+n}$
b $(p^m)^n = p^{mn}$
c $p^m\div p^n = p^{m-n}$ if $m>n$ or $1/p^{n-m}$ if $n>m$

The class may well remember that indices are not restricted to positive integers. The meanings given to fractional and to negative indices arise from a stipulation that these indices should obey the same rules as the positive integers. **b** can be used to find out what a fractional index means:

$(p^{\frac{1}{2}})^2 = p$ so that $p^{\frac{1}{2}}$ stands for $\sqrt{p}$.

Again, using 4 instead of p may give greater understanding. For the meaning of a negative index, rule **a** could be employed:

$$p^3 \times p^{-2} = p^{3-2} = p \quad \text{so that } p^{-2} = \frac{p}{p^3} = \frac{1}{p^2}$$

A negative index means a reciprocal.

Now rule **c** simply becomes $p^m \div p^n = p^{m-n}$, and allows a meaning to be given to p^0 if m is made equal to n:

$$p^m \div p^m = p^0 = 1.$$

Summarizing, the rules for dealing with indices are

a $p^m \times p^n = p^{m+n}$
b $(p^m)^n = p^{mn}$
c $p^m \div p^n = p^{m-n}$
d $p^{1/n} = \sqrt[n]{p}$
e $p^{-m} = 1/p^m$
f $p^0 = 1$

Examples will be needed to familiarize students with these processes.

Examples for section 1.1

1 What are the values of
a 9^2 **b** 9^{-2} **c** $9^{\frac{1}{2}}$ **d** $9^{-\frac{1}{2}}$?
[81, 1/81, 3, 1/3.]

2 Write the answers to the following as powers of 2
a $(8)^7$ **b** $\sqrt{8 \times 32}$ **c** $\sqrt[5]{4}$.
[2^{21}, 2^4, $2^{2/5}$.]

3 Write the answers to the following as numbers without indices
a $2^2 \div 2^{-1}$ **b** $3^0 \div 3^{-1}$ **c** $7^{-1} \times 14^2$.
[8, 3, 28.]

4 Which is the largest and which is the smallest of
$1/10^{-2}$, 2^6, $\sqrt[3]{1000}$?
[100, 64, 10.]

5 What is the value of a^2b^3/a^3b^2?
[b/a.]

1.2 Power of 10 notation – logarithms

In the Physics course both very large and very small numbers occur and these are expressed by using the power of 10 notation. Examples should be given to show the simplification which the method achieves:

$$100\,000\,000 = 10^8$$
$$250\,000\,000 = 2.5 \times 10^8.$$

Quite fearsome looking calculations can become tame when numbers are written out in this way, e.g.

$$\frac{3000\times 0.00018}{0.002}=\frac{3\times 10^3\times 1.8\times 10^{-4}}{2\times 10^{-3}}=\frac{3\times 1.8}{2}\times 10^2 = 2.7\times 10^2 = 270.$$

If fractional indices are introduced as well, the question, 'What is $10^{0.5}$?' arises. $10^{0.5}$ stands for $\sqrt{10}$ and is approximately equal to 3.162. It is not important that students should be able to calculate the square root of 10, but they should know that it can be done and that they can find its value from tables or by using a slide rule (see section 1.3).

$\sqrt{10}$ by the method of successive approximations

If students were interested – and only if – there might be some value in showing them the method of successive approximations.

$\sqrt{10}$ is obviously greater than 3 and less than 4 because $3^2 = 9$ and $4^2 = 16$.
Suppose it is $(3+x)$ where x is less than 1.
If $\sqrt{10} = 3+x$, then $10 = (3+x)^2 = 9+6x+x^2$.
x^2 will be a small number compared with the others and is neglected in comparison.

$$\therefore\ 10 \approx 9+6x \quad \text{so that } x \approx 0.17.$$

For a first approximation, $\sqrt{10} \approx 3.17$.
A second approximation can be obtained by letting $\sqrt{10} = (3.17+y)$ so that $10 = 3.17^2+6.34y+y^2$.
Again y^2 is neglected, giving $y \approx \dfrac{10-3.17^2}{6.34} = -\dfrac{0.0489}{6.34}$.

$$\therefore\ y \approx -0.0077.$$

Estimated value of $\sqrt{10}$ is 3.1623.
[Correct value 3.1624.]

Other fractional indices

Other fractional indices, such as $10^{0.75}$ and $10^{0.25}$, might be interpreted.

Since $10^{0.75} = 10^{\frac{3}{4}}$, it is the fourth root of 10^3 or the square root of the square root of 1000.

Since $10^{0.3} = 10^{3/10}$, it is the tenth root of 10^3 or that number which multiplied by itself 10 times gives a result of 1000. That number is very close to 2, for $2^{10} = 1024$.

The aim in introducing fractional indices is to show that other numbers can also be expressed as powers of 10, and then to plot a graph which will enable the index to be found for any number between 1 and 10.

Given that $10^{0.3} \approx 2.00$ and that $10^{0.5} \approx 3.16$, the class should be able to use the indices rules to supply the values of $10^{0.2}$, $10^{0.8}$, and then $10^{0.7}$. Then $10^{0.1}$ $(10^{0.8} \div 10^{0.7})$, $10^{0.4}$ $(10^{0.7} \div 10^{0.3})$, $10^{0.6}$ $(10^{0.3} \times 10^{0.3})$, and $10^{0.9}$ $(10^{0.7} \times 10^{0.2})$ follow.

Such calculated values are given in table 2:

x	0.0	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
$y = 10^x$	1.00	1.26	1.58	2.00	2.51	3.16	3.98	5.01	6.31	7.94

Table 2

If students are able to do so, they should plot a graph. x is called the logarithm of the number y to the base 10 – it is the power to which the base must be raised to give that number.

$$x = \lg_{10} y \text{ or simply } \lg y.$$

The example quoted at the start of this section could now be used to show logarithms in action, using the graph to obtain values:

$$\frac{3 \times 1.8}{2} \times 10^2 = \frac{10^{0.48} \times 10^{0.255}}{10^{0.3}} \times 10^2 = 10^{0.435} \times 10^2 = 2.7 \times 10^2 = 270.$$

The more complex operations of multiplication and division have been transformed into the simpler operations of addition and subtraction. Accuracy is limited if a graph is used, so tables of logarithms have been calculated. It may be of help if the above calculation is written out in a more conventional way – if only to revise how to obtain the logarithms of numbers greater than 10 and less than 1.

$$\begin{array}{rlrlr}
3000 = 3.0 \times 10^3 & \lg 3000 & = \lg 3.0 + \lg 10^3 & = 0.4771 + 3 = & 3.4771 \\
0.00018 = 1.8 \times 10^{-4} & \lg 0.00018 & = \lg 1.8 + \lg 10^{-4} & = 0.2553 - 4 = & \bar{4}.2553 \\
 & & \multicolumn{2}{r}{\lg (3000 \times 0.000\,18) =} & \bar{1}.7324 \\
0.002 = 2.0 \times 10^{-3} & \lg 0.002 & = \lg 2.0 + \lg 10^{-3} & = 0.3010 - 3 = & \bar{3}.3010 \\
 & & \multicolumn{2}{r}{\lg (3000 \times 0.00018 \div 0.002) =} & 2.4314
\end{array}$$

The number whose logarithm equals 2.4314 is 270.

There should be no difficulty with exercises involving $\lg p^x$ if students consider the meaning of p^x. They should see that $\lg p^x = x \lg p$. Some exercises should ensure that students can cope with use of logarithms in computation but it should not be overdone. It is more important that they should develop speed, confidence, and accuracy with a slide rule.

Examples for section 1.2

1 What are the values of the logarithms to base 10 of the following?
a 1000 **b** 10 **c** 0.1 **d** $10^{0.75}$ **e** 120 **f** 0.0275 **g** $5\sqrt{10}$.
[3.00, 1.00, $\bar{1}$.00, 0.75, 2.079, $\bar{2}$.439, 1.199.]

2 Which numbers have the following logarithms to base 10?
a 0 **b** 5 **c** 2.301 **d** $\bar{3}.959$ **e** 0.7782 **f** $\bar{5}.3010$.
[0, 100 000, 200, 0.009 1, 6.000, 2×10^{-5}.]

3 What is the logarithm to base 10 of the following product?
$10^{0.301} \times 10^{0.497}$.
[0.798.]

4 What are the values of the following?
a 5.31/2.12 **b** 5.31×2.12 **c** 63.1×2.72 **d** 6.31/27.2.
[2.505, 11.26, 171.6, 0.2319.]

5 Use logarithms to evaluate the following.
a 5^4 **b** $5^{\frac{1}{2}}$ **c** $\sqrt{7}$ **d** $\sqrt[3]{6}$ **e** $120^{0.3}$.
[625, 2.236, 2.646, 1.817, 4.206.]

6 What are the values of the following?
a 0.671/0.425 **b** 0.671×0.425 **c** $(0.325/0.526) \times 0.425$.
[1.579, 0.2852, 0.2626.]

7 The logarithm of a number to any base is the power by which the base must be raised to give that number. What are the values of the following?
a $\log_3 1$ **b** $\log_5 5$ **c** $\log_2 4$ **d** $\log_2 2^9$.
[0, 1, 2, 9.]

The book *Logarithms* by Austwick contains many worked examples and exercises.

1.3 The slide rule

Ability to use a slide rule with speed and accuracy will be of tremendous value to students, not only in the Physics course, but in any career involving numerical calculations. Some will need a lot of encouragement.

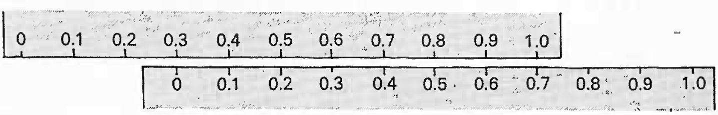


Figure 1

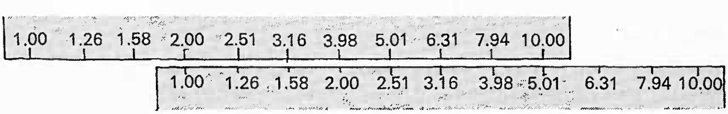


Figure 2

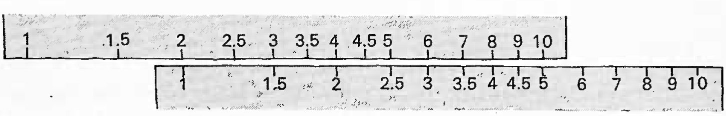


Figure 3

The use of the slide rule could be introduced by students making simple rules for themselves. This can conveniently be done using graph paper at least 200 mm long, which should be cut along a straight line to give two pieces with graph markings right up to the edge. A linear scale from 0 to 1.0 in divisions of 0.1 should be marked over a 200 mm length on each piece (see figure 1). When the 0 mark on one scale is placed against 0.3 on the other scale, then the number opposite 0.4 is 0.7. The arrangement obviously adds the two numbers, 0.3 and 0.4. To use two scales for multiplication, the process must be converted to addition. If the 0.3 on the top rule stood for $10^{0\cdot3}$, then opposite $10^{0\cdot4}$ on the lower scale, we find $10^{0\cdot7}$ which is the product of $10^{0\cdot3}$ and $10^{0\cdot4}$. Indeed, the scales could be marked with the values of 10^{0}, $10^{0\cdot1}$, $10^{0\cdot2}$, etc., as figure 2 shows. With the 1.0 on the lower scale set opposite 2.0 on the upper, all the numbers on the upper scale are just twice as big as the numbers on the lower scale underneath them.

The scales are obviously not very convenient as they stand – but the process for marking them more conveniently should be clear. Since $10^{0\cdot699} = 5.0$, on a 200 mm scale length, the 5.0 mark should be at a distance of $0.699 \times 200 = 139.8$ mm from the start of the scale, the 1 mark (figure 3).

Table 3 gives the distance of the principal scale markings along the scale:

Scale marking, *x*	1	2	3	4	5	6	7	8	9	10
lg *x*	0.000	0.301	0.477	0.602	0.699	0.778	0.845	0.903	0.954	1.000
Distance along scale /mm	0	60.2	95.4	120.4	139.8	155.6	169.0	180.6	190.8	200.0

Table 3

Other scale points can be inserted if desired.

Division can be accomplished by setting one number opposite the other and finding the result opposite the 1 or the 10 on the appropriate scale. Care is needed, in all cases, in placing the decimal point.

The scales referred to here are known as the C and D scales on commercial rules, C being on the slide and D on the lower stock. There are other scales on slide rules whose purpose should be described if pupils have rules to refer to. Many rules have A and B scales, A being on the upper stock and B on the slide. These scales are the squares of the C and D scales and enable squares and square roots to be found. For cubes and cube roots the K scale is used in conjunction with the C or D scale. CI and DI scales give the reciprocals of the C and D scales respectively.

Students should be given instruction and practice in the use of the C and D scales and in the evaluation of squares, square roots, and reciprocals. Efficiency in using a slide rule comes with experience and they should be encouraged to use the rule whenever possible. Detailed instructions for slide rule use are not given here. Most modern O-level texts for mathematics give these and manufacturers supply instructions too.

Exercises for section 1.3

1 Evaluate the following with the home-made slide rule:
a 3×2.5 **b** 4×5 (set the 10 mark under the 4) **c** 5/3 **d** 3/5 **e** 300/5 **f** 300/50.

2 Find the value of:
a $\sqrt{9}$ **b** $8^{\frac{1}{3}}$ **c** 1.5^3.

1.4 Scaling

Children grow up in a world of models – model cars, model aircraft, and model houses and furniture. These scale models are replicas, in miniature, of the real thing. Ask students to suppose they were to make a model of a house of the dimensions shown (figure 4), the length of the model to be 30 cm. What should its height and breadth be?

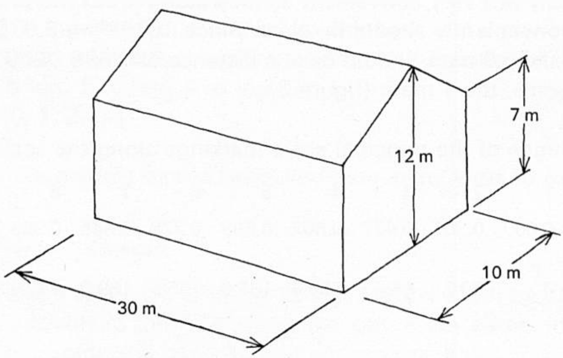


Figure 4

To make a replica, all linear dimensions must be reduced from metres to an equal number of centimetres. The scaling factor is 1/100 – the model is 1/100 of full scale – and all linear dimensions are reduced by the same *ratio*, the ratio of two quantities *of the same kind* being the number of times the first is bigger than the second. For example, the ratio of length to height of the house is 2.5, both for the real thing and for the model.

The scaling factor is not the same for all aspects of a model. Ask the class what the scaling factor is for the area of a wall ($1/10^4$) and for the volume of the house ($1/10^6$). Students should appreciate that: 1 like quantities are being compared again, *2* the scaling factor is different because areas involve products of two lengths and volumes products of three lengths, *3* it is not necessary to calculate areas or volumes to obtain the appropriate scaling factor. This is a good opportunity to remind the class of the formulae for the circumference ($2\pi r$) and the area (πr^2) of a circle, and for the surface area ($4\pi r^2$) and volume ($\frac{4}{3}\pi r^3$) of a sphere. These involve powers of the radius, so that, if the radius is scaled by a factor k, the area scaling factor will be k^2 and the volume scaling factor k^3.

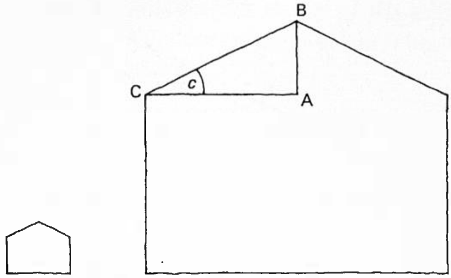


Figure 5

In scaling, angles do not change. The end of the house in the real case is a similar figure to the end of the model house (figure 5). All the corresponding angles are equal. Students may need to be reminded of the trigonometrical ratios. In a right-angled triangle, e.g. $\triangle ABC$,

the **sine** of an angle $= \dfrac{\text{opposite side}}{\text{hypotenuse}}$ i.e. $\sin c = \dfrac{AB}{BC}$

the **cosine** of an angle $= \dfrac{\text{adjacent side}}{\text{hypotenuse}}$ i.e. $\cos c = \dfrac{AC}{BC}$

the **tangent** of an angle $= \dfrac{\text{opposite side}}{\text{adjacent side}}$ i.e. $\tan c = \dfrac{AB}{AC}$

All these are ratios. In scaling, both lengths will change by the same factor and the ratio does not change. So the angles remain the same.

A few simple scaling exercises will be needed here. Chapter 4 of PSSC *Physics* is useful reading for students, with examples at the end. There is a film to go with this reading if teachers care to use it. ('Change of scale', 900 4120–6, Guild Sound and Vision Ltd, formerly Sound Services Ltd).

The meanings of the sine, the cosine, and the tangent of an angle must be learned, for they are part of the daily language of science and mathematics. Examples may be needed to help learning, but these should not be long tedious calculations involving difficult angles. Rather, they should be directed towards recognition of the appropriate ratio and obtaining a value from tables.

Students might be expected to 'discover' some trigonometrical relationships for themselves by doing questions of a more structured kind, such as 10, 11, and 12 in the examples which follow. They should know the relationships between the ratios for angle θ and those for angle $(90-\theta)$, and they might benefit from a first look at the approximations which can be applied when angles become small. The trigonometrical form of Pythagoras' theorem is not difficult to obtain, but the

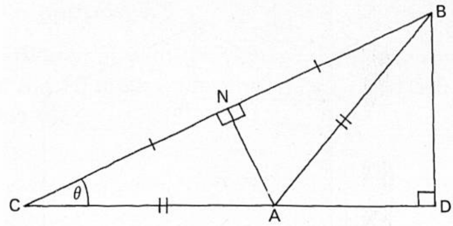


Figure 6

example that follows, leading to expressions for the sine and cosine of 2θ should only be used as a 'buffer' for the fastest pupils at this stage.

In figure 6, ABC is an isoceles triangle with AB = AC = 1 unit. AN is drawn perpendicular to BC, and because ABC is an isosceles triangle, it bisects BC, i.e. BN = NC so that NC = $\frac{1}{2}$BC. BD is drawn perpendicular to CA produced. Let angle BCA = θ.

- **a** What is the size of angle CBA?
  [θ.]
- **b** What is the size of angle BAD?
  [2θ.]
- **c** What is the size of $\sin 2\theta$?
  [BD.]
- **d** Use triangle BDC to obtain a value for $\sin\theta$.
  [BD/BC.]
- **e** Use triangle ANC to obtain a value for $\cos\theta$.
  [NC.]
- **f** What is the product $\sin\theta \times \cos\theta$ equal to?
  [$\frac{1}{2}$BD.]
- **g** Use the answer to **c** to obtain an expression for $\sin 2\theta$.
  [$2\sin\theta\cos\theta$.]
- **h** Use the fact that AD = CD − 1 to find out what $\cos 2\theta$ equals.
  [$2\cos^2\theta - 1$.]

Examples for section 1.4

1 The data in table 4 refer to two 'model' cars.

	Length	Breadth
Rolls-Royce Silver Shadow	5.2 m	1.8 m
Rolls-Royce model	76 mm	28 mm
Aston-Martin DB6	4.6 m	1.7 m
Aston-Martin model	73 mm	27 mm

Table 4

Which is the scale model and what is the scaling factor?
[Aston-Martin, 1/63 approximately.]

2 A geographers' globe is a scale model of the Earth. If the radius of the globe is k times the radius of the Earth, what is the ratio of the following quantities on the globe to those on the Earth?

a the length of the Equator

b the area of the circle of which the Equator is the circumference

c the surface area

d the volume

[k, k^2, k^2, k^3.]

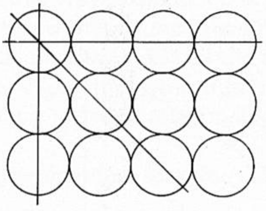


Figure 7

3 Figure 7 is a plan view of a scaled up version of the packing of atoms in a simple crystal. Lengths have been scaled up by a factor of 10 million. What is the scaling factor for **a** the area occupied by an atom **b** the volume occupied by an atom? **c** Are the angles between the places joining atom centres different from those drawn on the scaled up diagram?

[10^{14}, 10^{21}, No.]

4 If all the atoms in a 22 s.w.g. copper wire grew to be 50 mm in diameter, what would be the diameter of the wire?
Diameter of a copper atom = 2.5×10^{-10} m. Diameter of 22 s.w.g. wire = 0.711 mm.

[140 km approximately.]

5 If all the linear dimensions of a wire are scaled down by a factor of 4, by what factor does **a** the volume **b** the surface area change?
c How would the breaking strength change if the strength were proportional to the cross-sectional area?

[1/64, 1/16, 1/16.]

6 The heat loss per unit time from a living body is proportional to the surface area of the body. If the linear dimensions of a human being were doubled, how would the heat losses change? By what factor do you think the food intake would change?

[4.]

7 If you make a one-tenth model of a table out of the same material, and the original had a mass of 20 kg, what would be the mass of the model?

[20 g.]

8 A rocket rises vertically for 15 km, then at 30° to the vertical for 80 km along its path, and finally at 60° to the vertical for a further 100 km. How high is it above ground level and how far has it moved horizontally?

[134 km, 127 km.]

9 A trolley on a horizontal surface is pulled with a force of 3N by means of a string inclined at 40° to the horizontal. What is the force accelerating the trolley along the plane?

[2.3 N.]

10 ABC is a right-angled triangle and the angle at C is equal to θ (figure 8).

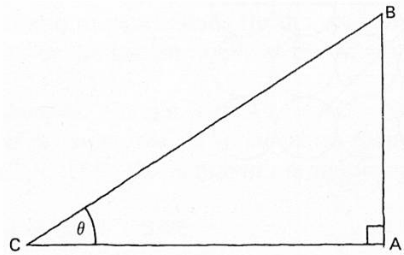


Figure 8

a Write down the ratios for $\sin\theta$, $\cos\theta$, $\tan\theta$.
b What is the size of angle B in terms of θ?
c Write down the ratios for $\sin(90-\theta)$, $\cos(90-\theta)$, $\tan(90-\theta)$.
d What does $\sin(90-\theta)$ equal amongst the answers to **a**?
e What does $\cos(90-\theta)$ equal amongst the answers to **a**?
f Express $\tan(90-\theta)$ in terms of $\tan\theta$.

11 Using the triangle in **10**, Pythagoras' theorem tells us that $AB^2+AC^2 = BC^2$.
a Find an expression for AB in terms of BC and θ.
b What is AB^2 equal to?
c What is AC^2 equal to in terms of BC and θ?
d Substitute these in the expression above and simplify the resulting equation.

12 Draw a right-angled triangle having one angle very small (5° or less). Measure the angle and work out the approximate values of the sine, cosine, and tangent of this angle. Examine tables for the ratios for small angles. What can you say about the values of sine, cosine, and tangent for small angles?

Section 2

Functions

There is much in Section 2 that is revision of earlier work, and teachers should not spend long on it if the class is familiar with and able to handle the content. The work is concerned with displaying and recognizing the functional relationships $y \propto x$, $y \propto 1/x$, $y \propto x^2$, $y \propto 1/x^2$, most of which appear in the Physics course at some stage. In this way, experience is gained with the graphical and algebraic methods of expressing these functions or patterns. Those students who are not mathematically inclined will need all the reinforcement of basic ideas that is possible. Plenty of experience with numbers, graphs, and equations will help in establishing a feeling for such relationships.

Graphs of the trigonometrical functions and growth and decay patterns are dealt with later.

2.1 Introduction

A brief introduction will be needed. It might be on the following lines:

'There are very many examples in physics, in other sciences, in economics, and in daily life, where changing one thing can result in the change of another thing. To take some examples, the temperature at which water boils changes if the external pressure changes; increasing the force applied to a rubber band fixed at one end causes an increase in its length; increasing the price of motor fuel causes an increase in the cost of groceries. If more is to be found out about the detail of the processes involved, or if predictions regarding behaviour are to be made, or if different materials are to be compared, measurements must be made to obtain sets of numbers which express the value of one quantity for a chosen value of some other quantity. As an example, in studying the way in which a spring extends, we might apply known forces and measure the extensions those forces produced.

Force, F/N	0	2	4	6	8	10
Extension, e/mm	0	3	6	9	12	15

On the top line are the numbers of units of force used, the factor whose magnitude the experimenter selects. Below each of those numbers is a number expressing the extension resulting from the application of that particular force. The quantity whose value is chosen is frequently called the independent variable (in this case, the force), the other quantity being the dependent variable. Clearly, there is a pattern between these pairs of quantities. What is it and what would be a useful way of displaying it?'

It is probably enough to say that a function is a relationship between one set of quantities and another. Students who have followed a course based on one of the recent mathematics projects may draw a mapping diagram to show the relationship (figure 9). Because there is a linear relationship between force and extension, equal increments of force give equal changes in the extension.

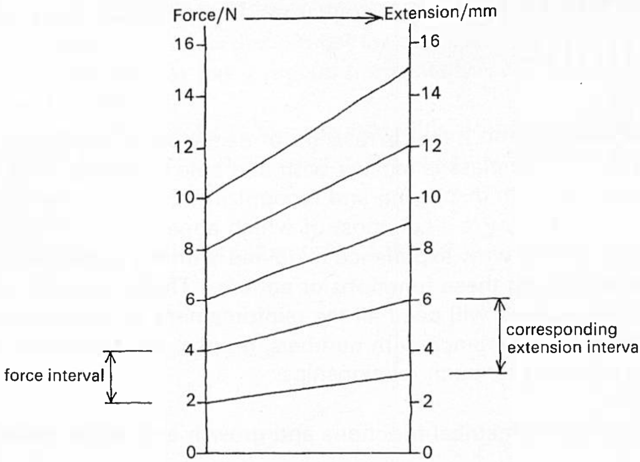


Figure 9

In physics, another way of displaying such patterns is of enormous value. This is the graph.

2.2 Co-ordinates

If necessary (and it should usually not be), the idea and application of a co-ordinate system can be revised by means of the National Grid system used on Ordnance Survey maps. Students could practise identification of places on local maps by means of co-ordinates and should appreciate that, to avoid confusion, it is necessary to have a rule about which co-ordinate is stated first. That rule – the co-ordinate for distance east of the origin before that for distance north of the origin – is the same as the one used when specifying position relative to a set of cartesian co-ordinate axes, the 'east' reading being referred to as the x co-ordinate, the 'north' reading as the y co-ordinate. If needed, exercises should be set to give practice in plotting points, and negative co-ordinates should be included in the data. It might be wise to include an example in which the axes represent quantities which are not distances.

Exercise for section 2.2

1 Draw axes on a sheet of graph paper. Choose the origin to be somewhere near the middle of the paper. Mark values along the axes from 0 to 10. Plot the following points on the graph paper:

$(0, 0), (4, 3), (3, 4), (7, 8), (0, 7), (-3, 0), (-4, -3)$.

Where does the line joining $(-3, 0)$ with $(7, 8)$ cut the line joining $(3, 4)$ with $(0, 7)$?

[At (2.6, 4.4).]

2.3 Proportionality

Descriptions of the graph representing the data given in section 2.1 should be asked for and the following points brought out:

1 The graph is a straight line passing through the origin.
2 It represents a behaviour in which doubling the quantity represented by the *x* co-ordinate doubles that represented by the *y* co-ordinate, i.e. if (a, b) lies on the line, then $(2a, 2b)$ is also on the line.

Students should be told that this is called 'direct proportionality', that it is written as *force* $\propto$ *extension it produces*, and that there are many examples of pairs of quantities which behave similarly. Such behaviour is not confined to scientific matters: the cost of buying sweets is in direct proportion to the weight bought, and if the weight bought is doubled, the cost is doubled too!

A few exercises in which data are examined to see if one quantity is proportional to another would be useful here. Examples should progress from the easy to the more difficult and should include cases in which direct proportionality does not occur. A sample of exercises appears below.

Examples for section 2.3

1 Is the distance travelled by these cars proportional to the time taken?

a

Time/s	0	40	80	120	160	200
Distance/km	0	1	2	3	4	5

[Yes.]

b

Time/s	0	20	45	70	105	135
Distance/km	0	1.6	2.6	3.3	4.1	4.7

[No.]

2 Is *y* proportional to *x* for the following data?

a

x:	1.7	2.8	3.4	4.2	5.3
y:	5.1	8.4	10.2	12.6	15.9

[Yes.]

b

x:	3.0	5.5	7.5	10.0
y:	8.0	18.0	26.0	36.0

[No.]

c

x:	20.0	22.5	24.7	27.3	28.9
y:	1.00	1.07	1.19	1.44	1.76

[No.]

2.4 Proportionality expressed algebraically

If x represents the value of the independent variable and y that of the dependent variable, then, if $y \propto x$, the data will show that y/x has a constant value for all the values of x. Perhaps the class will have noticed this already, but it should be verified for the data given in the examples used. If such a relationship between the quantities involved is known, it has advantages over the pictorial methods of displaying data; it is briefer and easier to use, and is perhaps more accurate.

Generalizing further, a pattern of direct proportionality can be written as $y/x = m$ where m is the constant of proportionality. Some students may find the following helpful in appreciating the significance of m. In the example chosen in section 2.1, each value of the dependent variable corresponded to just one value of the independent variable – as though there were a machine (figure 10) into which one fed the value of the x co-ordinate, and the machine multiplied this by m and fed the result to the output.

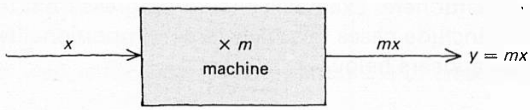


Figure 10

Some practice in manipulating the equation and writing it in different forms may be needed (e.g. $y/x = m$, $y = mx$, $x = y/m$), and students should know how to get the value of m from the graph. Some discussion of the units of m will be necessary, including the case where x and y have the same units and m is simply a number. When determining the value of m from the graph, it is wise to point out that it is necessary to choose *two* points on the straight line, one of them, in this case, being the origin.

The introduction of algebraic symbols may be a difficult step for some students. The step may be made easier by using symbols, other than x and y, to represent physical quantities, and by frequent statement of the meaning of those symbols. Numerical examples will help the weaker mathematicians to gain confidence, particularly in obtaining equations to represent numerical data and in solving simple direct proportion problems.

Examples for section 2.4

1 Given that $y \propto x$, fill in the blanks below.

x:	12	10	7	...	...	-2
y:	42	...	...	14	0	...

Write an equation relating x to y.

[35, 24.5, 4, 0, -7; $y = 3.5x$.]

2 The mass m, of a substance is directly proportional to its volume V. It is found that a cube of side 5 cm has a mass of 1 kg. Write an equation for m in terms of V. What is the unit of the constant and what name is usually given to it? Rewrite the equation with the constant as the subject.

[$m = 8V$ using g and cm^3 units, *or* $m = 8000V$ using kg and m^3 units. Density $= m/V$.]

3 Write equations for those patterns in the examples for section 2.3 in which one variable is proportional to the other.

[For **1a** distance in km $= \frac{1}{40} \times$ time in s. For **2a** $y = 3x$.]

4 The following readings were taken in an experiment to measure how much current I flowed through a wire when a p.d. V was placed across it.

I/A:	1.25	2.15	3.15	4.35
V/V:	1.40	2.35	3.45	4.75

Is this an example of direct proportionality? If you think it is, write down an equation for V in terms of I, stating the unit of any constant used.

[Yes; $V = 1.09\ I$; VA^{-1} or Ω].

2.5 Inverse proportion

Inverse proportion should also be examined and represented by equations. A simple pattern of this kind is:

x:	1	2	3	4	5	6
y:	120	60	40	30	24	?

This pattern is clearly not one of direct proportionality, for as x gets larger, y decreases. What happens when x is doubled? What is the missing number in the y sequence? Can a way of getting a constant number from corresponding values of x and y be discovered?

It may help to remind the class of a Boyle's Law experiment (Nuffield O-level Physics *Guide to experiments IV*, experiment 76) for which the following is a typical set of values:

Air pressure, p/kN m^{-2}	100	120	140	160	180	200
Air volume, V/cm^3	60.0	50.0	42.9	37.5	33.3	30.0

The fact that pV is constant may be more immediately obvious than that $p \propto 1/V$, so that the pattern will be expressed as $pV = k$. This should also be written as $V = k/p$. The volume is proportional to the reciprocal of the pressure, a behaviour described as inverse proportion. Students should also plot a graph to show how V varies with p and see that:

it is not a straight line,
it does not pass through the origin,
if (a, b) lies on the curve, then $(2a, b/2)$ and $(a/2, 2b)$ do also.

Some exercises should follow this, each including some discussion of the units of the constant involved.

Examples for section 2.5

1 Suppose that the amount by which the length of a block of material is compressed for a given force depends on the cross-sectional area as follows:

Cross-sectional area/mm²	100	200	300	400
Amount compressed/mm	6	3	2	1.5

Is this inverse proportion? What equation would represent this pattern? What would be the value and the unit of the constant?

[Yes; compression × area = 600 mm³.]

2 Suppose that the frequency of the note emitted by a taut string plucked at its midpoint varies with the length of the string as follows:

Length/mm	500	400	332.5	250
Frequency/Hz	256	320	384	512

Is this inverse proportion? What is the equation these quantities fit, and what is the unit of the constant?

[Yes; length × frequency = 128 m s^{-1}.]

3 Do the following numbers fit an inverse proportion pattern?

p:	3.1	2.7	2.2
q:	1.9	2.15	2.6

[No.]

4 The time of swing of a pendulum of fixed length is inversely proportional to the square root of the acceleration due to gravity. A 1-second pendulum is transported to the Moon where the acceleration due to gravity is only $\frac{1}{6}$ of the Earth value. What is the time of swing?

[2.45 s.]

2.6 More difficult patterns or relationships

Experience should be extended to other patterns. This time, work could start from the algebraic equation, a pattern being obtained from it and a graph plotted to display that pattern. The relationships $q = kp^2$ and $q = kp^{-2}$ might be considered, k being given a definite numerical value.

Students should be clear that q is not directly or inversely proportional to p in these cases, but that q is so related to p^2. If there is any doubt, numerical work will reveal the proportionality and graphs of q against p^2 and q against $1/p^2$ respectively will be straight lines through the origin. The effect of the value of k on the pattern and on the graphs may also need some discussion.

This work should not be laboured. The important thing is that more complicated patterns exist. They give an opportunity for more practice in handling physical quantities, and their arithmetical and graphical expression.

Examples for section 2.6

1 What is the algebraic relationship for each of the following sets of numbers or quantities?

a

p:	0	1	4	9	16
q:	0	1	2	3	4

[$q = p^{\frac{1}{2}}$ or $q^2 = p$.]

b

p:	0	1	2	3	4
q:	0	2	8	18	32

[$q = 2p^2$.]

c

Time of swing of pendulum/s	2	4	6
Length of pendulum/m	1	4	9

[Time $= 2 \times \sqrt{\text{length}}$.]

2 The time of swing of a pendulum is proportional to the square root of its length. If a pendulum of length 0.25 m has a time 1 s, how long should it be to have a time of 2 s?

[1 m.]

3 In successive seconds, a falling object travels 5, 15, 25 metres. How is the distance it has travelled from its starting point related to the time it has been travelling?

[t/s:	0	1	2	3	
d/m:	0	5	20	45	$d = 5t^2$.]

4 The illumination of a surface by a certain lamp is in inverse proportion to the square of the distance of the surface from the lamp. Where would you place the surface so that its illumination is twice as big as that at 1 m from the lamp?

[0.707 m.]

2.7 Dependence on more than one variable

Very often a quantity is found to depend on more than one variable. For example, the time, t, taken to travel at steady speed from one place to another, depends directly on their distance apart, d.

That is, $t \propto d$ if the speed is constant.

However, the time to travel between two places a fixed distance apart at steady speed is in inverse proportion to that speed, v.

That is, $t \propto 1/v$ if d is constant.

How will t change if both d and v change? If both d and v are doubled, how does t change? If d is doubled and v is halved how does t change? How has d/v changed?

Students should see that $t \propto d/v$ or $t = kd/v$.

Some teachers may care to supplement the work at this stage with experiments leading to simple quantity patterns, e.g. distance–time measurements for a trolley rolling down an incline (Nuffield O-level Physics *Guide to experiments IV*, experiment 3), but they should consider whether or not the time involved could not be spent better in giving students practice at manipulating 'fictitious' patterns. Confidence increases when the class handles numbers successfully, algebraic equations mean more when they are seen to represent patterns relating quantities which are given actual values, and the weaker members of a class may benefit by having more practice.

Examples for section 2.7

1 The electrical resistance of a material in the form of a wire varies directly as its length and inversely as its cross-sectional area. Write down an algebraic expression for its resistance.

$$\left[R = \frac{kL}{A}.\right]$$

2 The pressure of a gas is thought to be inversely proportional to its volume when the temperature is kept constant, and directly proportional to its absolute temperature when the volume is kept constant. Do the following values support that view?

Pressure/N cm^{-2}	1000	1200	1500
Volume/cm^3	72	70	60
Temperature/K	300	350	375

$$\left[\text{Yes; } \frac{pV}{T} = 240 \text{ N cm K}^{-1}.\right]$$

Section 3

Linear graphs

3.1 Introduction

The work of Section 2 has shown that if y is proportional to x, that is, if $y = mx$, and if x can take on any value (if necessary, within a limited range), then a graph of y against x is a straight line passing through the origin. The work of Section 3 could start by asking what would be the effect if m had a different value. No doubt, it will be easier for some if there are numerical data to work from. Distance–time relationships for constant speeds provide suitable material and students might start by plotting graphs to represent $y = 3x$ and $y = 5x$ on the same set of axes, with y representing distance travelled and x the time elapsed. The graph with the larger value of m is steeper.

3.2 Measuring the slope or gradient

None of what immediately follows is very surprising – in fact, some will regard it as obvious and hardly worth attention. Nevertheless, it is worth pursuing for the sake of the more difficult work to come. It would be wise to establish the method for measuring the slope or gradient of the graph (figure 11) from two points, P and Q, marked on the line. Students should complete the right-angled triangle PQR, measure the 'lengths' of PR and QR from the scales marked on the axes, and calculate the ratio QR/PR. Of course, this has the same value as m in $y = mx$. They should also be shown this more generally in terms of the co-ordinates of P, say (x_1, y_1), and Q (x_2, y_2). Then PR $= (x_2 - x_1)$, QR $= (y_2 - y_1)$,

$$\text{and the gradient} = \frac{y_2 - y_1}{x_2 - x_1}.$$

But if $y = mx$, then $y_2 = mx_2$ and $y_1 = mx_1$,

$$\text{so that the gradient} = \frac{mx_2 - mx_1}{x_2 - x_1} = \frac{m(x_2 - x_1)}{(x_2 - x_1)} = m.$$

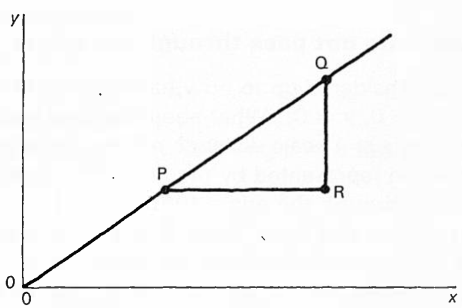
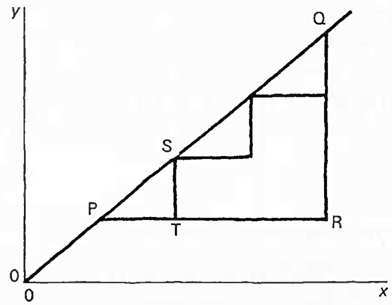


Figure 11

There are some important points to drive home about this. First, it doesn't matter which two points on the line are chosen – except that it would be better to choose points widely separated to make the measurement as accurate as possible. If students find the algebraic approach difficult, they might appreciate the geometrical method illustrated in figure 12, or prefer to show it by basing calculations on different points. Second, the visual steepness of the line depends on how the scales are marked off on the axes, but the gradient does not, because PR and QR are measured from the scales on the axes.



PR = 3PT
QR = 3ST
QR/PR=ST/PT

Figure 12

For the sake of the work to come, teachers should take the opportunity to clarify the meaning of $m = (y_2 - y_1)/(x_2 - x_1)$. $(y_2 - y_1)$ is the change of y corresponding to a change in the value of x equal to $(x_2 - x_1)$. Thus the gradient measures the rate at which y changes with respect to x.

Exercise for section 3.2

Express the following physical quantities as 'the rate at which —— changes with respect to ——.'

a speed **b** acceleration **c** the force constant for a spring **d** resistance.

[**a** distance, time; **b** velocity, time; **c** force, extension; **d** p.d., current.]

3.3 Lines which do not pass through the origin

All the linear graphs considered up to now have passed through the origin (if $y = mx$, then when $x = 0$, $y = 0$). What about the line which does not, the line which crosses the y-axis at a scale distance c from the origin, the intercept? What is the algebraic expression represented by the line? The class might be asked to imagine a parallel line through the origin (figure 13). The two lines clearly have the same gradient and for any and every value of x, the corresponding value of y on the line with intercept c is just c greater than the value of y on the line through the origin. The equation for the latter is $y = mx$, whilst that for the former is $y = mx + c$. A quick look at numerical examples, e.g. plots of $y = 5x$ and $y = 5x + 7$, should convince students of this:

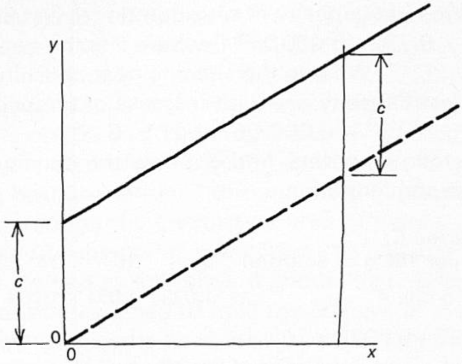


Figure 13

For x values of	0	1	2	3	4	5
$y = 5x$ has y values:	0	5	10	15	20	25
$y = 5x+7$ has y values:	7	12	17	22	27	32

The graphs are parallel lines; the gradients measured from two points on each line are the same.

It is important that the class should note that, if $y = 5x+7$, y is *not* proportional to x, for doubling the value of x does *not* give a y value twice as big. In general terms, if $y = mx+c$, then when $x = 1$, $y = m+c$, and when $x = 2$, $y = 2m+c$, but $2m+c$ is not equal to $2(m+c)$. Thus, $y = mx+c$ gives a straight line graph (for this reason it is called a linear relationship) which has a gradient of m and an intercept c. When $c = 0$, the graph goes through the origin and y is proportional to x. Changes of scale by which the axes are marked out do not change the values of m or c.

Students will need practice in measuring gradients and intercepts. Equations of lines parallel to the axes should not be omitted ('If we think of the line $x = 2$, y can have any value you like but x must always equal 2'). Negative intercepts and negative gradients must be included too. In every case, the equation plotted should also be rearranged with y as the subject, i.e. as $y = mx+c$. A few examples follow.

Examples for section 3.3

1 Sketch the graphs of:
a $y = 3x$ **b** $y = 3x+4$ **c** $y = 3x-4$ **d** $y = 4-3x$ **e** $x = 3y$ **f** $x = 3y-4$.
What are the gradient and intercept in each case?

2 The following data show how the length of a rod depends on the temperature measured in °C.

Temperature θ/°C	50	100	150	200
Length L/mm	1001	1002	1003	1004

What is the length at 0 °C? What is the gradient of the graph? What is the equation of the graph?

[1000 mm; 2×10^{-2} mm °C^{-1}; $L = 2\times10^{-2}\theta+1000$.]

3 The specific heat capacity of nitrogen under certain conditions can be represented by $C = 0.732+0.000067\ \theta$ where θ is the temperature in °C and C is measured in J g^{-1} °C^{-1}. What is the specific heat capacity at 0 °C? How rapidly does the specific heat capacity rise with increase of temperature?

[0.732 J g^{-1} °C^{-1}; 0.000 067 J g^{-1} °C^{-2}.]

4 Using the following data, find out how the density of sodium chloride solution varies with concentration.

Concentration *C*/ grammes per 100 g of solution	5	10	15	20	25
Density *d*/g cm^{-3}	1.033	1.071	1.109	1.147	1.185

[$d = 0.995+0.0076\ c$.]

5 Given that x and y are related linearly and that when $x = 1$, $y = 1$ and when $x = 7$, $y = 4$, find the relationship.

[$2y = x+1$.]

3.4 The linear graph in scientific work

This is an appropriate point at which to say something about experimental work and straight line graphs. In scientific work, it is frequently necessary to find out how one quantity depends on another. This can be done by taking corresponding measurements of the two quantities and plotting a graph. If the graph is linear, then it should be easy to find that relationship. If it is not linear, the puzzle is harder. There is, however, a small though important difference between what has been done up to now and what has to be done with experimental measurements. Up to now an equation has been taken (or fictitious 'results' derived from an equation have been used), x has been given an appropriate value, and the value of y has been calculated. Provided the arithmetic was faultless, the value of y was known exactly and the various points, if plotted correctly, lay exactly on a straight line. However, with experimental measurements, there will inevitably be some error in those measurements and the points when plotted will, in all probability, not lie exactly on a straight line. The first decision needed is whether or not the relationship can reasonably be represented by a linear graph. Consider the examples shown in figure 14.

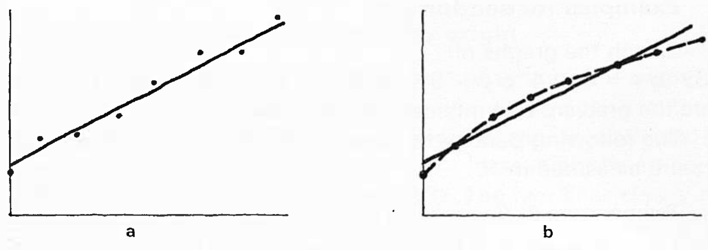


Figure 14

In figure 14*a* the points lie scattered about the line without any definite trend one way or the other. In figure 14*b*, however, though the distance of the points from the line is no greater than in figure 14*a*, there is a definite trend. In each case there are as many points below the line as there are above, and in the case of figure 14*b* there are even 2 points on the line, but the probability would be that the relationship between x and y indicated by figure 14*b* would be better represented by a curve. Drawing 'the best straight line' through a set of points does not necessarily mean the line passing through the greatest number of them – but a line such that there is no definite trend of the points away from the line and with about as many points on one side of the line as the other. It should be pointed out that linear relationships should not be assumed to hold over ranges of the variables greater than those in which measurements have been made, unless there are good reasons for assuming this.

Students should now be given examples of experimental measurements which might follow linear laws and be asked to find the best relationship between them. Some of these examples should involve the use of a 'false' origin and the problem of finding the intercept. Though the direct method (measure the gradient and then substitute the co-ordinates of a point on the line into $y = mx+c$ in order to calculate c) may appeal, the opportunity can be taken to show the following method.

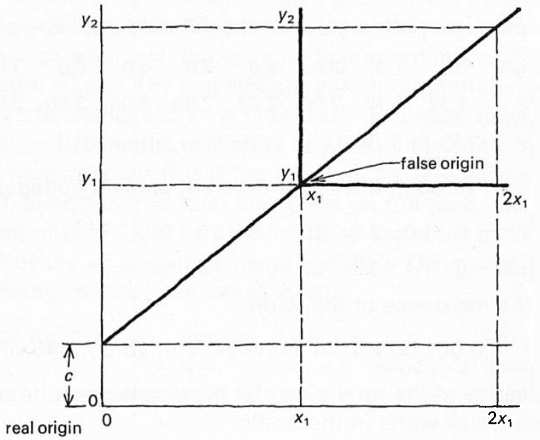


Figure 15

By similar triangles (figure 15):

$$\frac{y_2-y_1}{y_1-c} = \frac{2x_1-x_1}{x_1} = 1.$$

Thus, $y_2-y_1 = y_1-c$, giving $c = y_1-(y_2-y_1)$ which can be obtained from the graph with the false origin, though it depends on having an x scale which includes a value twice that of the smallest.

Examples for section 3.4

Are there linear relationships between the corresponding quantities in the following experimental results? If so, establish the equation linking them.

1 The density of water at different temperatures:

Temperature/°C	10	20	30	40	50
Density/kg m^{-3}	999.7	998.2	995.6	992.2	988.0

[This relationship is not linear.]

2 The deflection of the free end of a cantilevered beam for different loads at the free end:

Load/N	0.5	1.0	1.5	2.0	2.5	3.0
Deflection/mm	31	58	89	121	149	181

[Linear: deflection in mm $= 60 \times$ load.]

3 The depth of immersion of a loaded test-tube for different loads:

Load/10^{-2} N	0	2.0	3.0	5.0	6.0	8.0	10.0
Depth/mm	1	6	10	12.5	17	21	26

[Linear: depth in mm $= 250 \times \text{load} + 1$, with load in N.]

4 The current, I, through a solution for different applied voltages, V:

Current/mA	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0
Voltage/V	1.32	1.72	2.00	2.22	2.69	3.08	3.35	3.66	3.96

[Linear: $V = 340\,I + 1.30$ (V in volts, I in amperes).]

5 The current I through a wire for different applied voltages V:

Voltage/V	0	1.0	2.0	3.0	4.0	5.0	6.0
Current/mA	0	1.4	3.3	4.7	6.1	7.9	9.2

What is the resistance of the wire?

[Linear: $I = 0.001\,55\,V$ (I in amperes, V in volts), 645 ohms.]

6 The frequency of the note given by blowing across the top of an open bottle and the volume of water in the bottle:

Frequency/Hz	320	341	384	427	480	512
Volume/cm^3	33.5	51.5	79.0	105	123	131

[Not linear.]

3.5 The area 'under' a graph

The aim here is to give a first introduction to finding the area under a graph by the method of strip-division and to ascertaining whether that area has any physical significance. Students should have met this in their O-level work and it may have been raised earlier when work on linear graphs started.

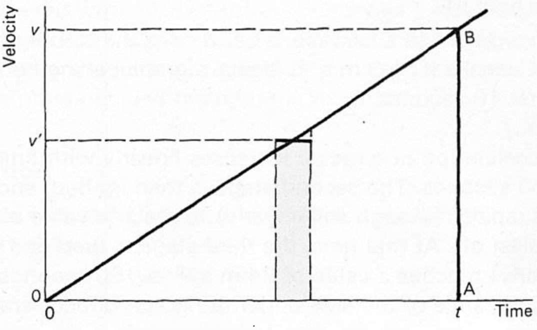


Figure 16

Sometimes the 'area under a graph' has a physical significance and ~~the~~ class should be shown that the area under a velocity–time graph gives the distance travelled. This is quite simple for the case of uniform velocity. For an object travelling at constant acceleration from rest, the velocity is proportional to time and the velocity–time graph (figure 16) is a straight line through the origin. Now consider the moment when its velocity is v'. A very small time later, its velocity will have increased a small amount and it will have travelled a distance equal to *average speed* × *time interval*. This is the same as the area of the shaded strip if the height and width are measured from the scales on the axes. The total distance travelled from rest in time t will be the sum of all the strips making up $\triangle$OAB – and the 'line of tops' of the strips can be made to follow OB more closely by simply imagining very many strips of very small width.

$$\text{Distance travelled, } s = \text{area of } \triangle \text{OAB}$$
$$= \tfrac{1}{2}vt \quad \text{where } v = \text{velocity at time } t.$$

But if the acceleration is a, then $v = at$ and $s = \frac{1}{2}at^2$.

A similar argument shows that the area under a force–extension graph for a spring gives the energy transformed in stretching it slowly and that the area under a p.d.–charge graph for a capacitor gives the electrical energy stored.

It is important to emphasize that, in computing these areas, the dimensions must be taken from the scales marked on the axes. It is worth mentioning that it is, of course, only the area 'under' the graph if the variables are plotted the 'right' way round. Plotted the other way round, it would be the area to the left of the line which was significant.

Examples for section 3.5

1 Data relating to the stretching of a spring are given in section 2.1. How much energy is transformed in stretching it slowly

a from an extension of 0 mm to an extension of 5 mm?

[8.3×10^{-3} J.]

b from an extension of 5 mm to an extension of 10 mm?

[25×10^{-3} J.]

2 In a flying start to a car race, a car passes the starting line at 30 m s^{-1} with a constant acceleration 3 m s^{-2}. Using a graphical method, find how far it travels in the first 10 seconds.

[450 m.]

3 The acceleration of a rocket increases linearly with time and has reached 2 m s^{-2} after 60 seconds. The second stage is then ignited, and the acceleration increases more rapidly (though still linearly) to reach a value of 8 m s^{-2} at 120 seconds after blast off. At that time, the third stage is fired and the acceleration (still increasing linearly) reaches a value of 24 m s^{-2} at 160 seconds after blast off. What is the significance of the area under the graph of acceleration against time and what is the value of the quantity after 160 seconds?

[Velocity, 1000 m s^{-1}.]

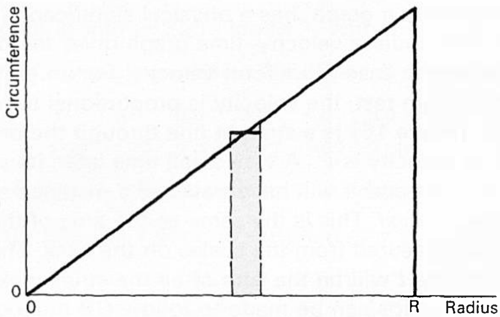


Figure 17

4 The circumference of a circle is proportional to the diameter, the constant being π, and a graph of circumference against radius (figure 17) is a straight line passing through the origin and with a gradient of 2π. What does the area of the narrow shaded strip represent? What is the area under the graph up to radius R equal to?

[πR^2.]

Section 4

Non-linear graphs

Section 4 is concerned with some simple non-linear functions and raises the question of how they might be plotted to obtain straight lines. Students who have already been through the work of Section 2 ('Functions') will have had much of the experience of graph work suggested in the Introduction that follows and for them it need be no more than a brief recapitulation. For those who omitted the work of Section 2 a more detailed treatment is worth while.

4.1 Introduction

The work could start with the class plotting and deducing the shapes of graphs described by algebraic equations, e.g.:

1 $y = 4x$	*2* $y = 4x-2$	*3* $y = 4x^2$	*4* $y^2 = 4x$
5 $y = 4/x$	*6* $y = 1/x$	*7* $y = 4/x^2$	*8* $y = 4(x-2)^2$

It will probably be enough if each student tackles two examples and is asked to describe results to the other members of the class, pointing out the important features of each graph, e.g. in the case of $y = 1/x$, y gets larger as x gets smaller and vice versa, and the curve never crosses the axis. It will be necessary to prompt students to think about both positive and negative values of the variables. The significance of constant factors should be considered and so should the effects that a change of magnitude or sign would have.

It is not intended that any of this should become laboured. The aim is to get some feel for graphical work, not that the class be given a potted course on algebraic geometry. Teachers should limit or extend the work according to their students' abilities.

The forms of the equations above are the same as those to be met within the Physics course, and some examples drawn from physics follow. Teachers may prefer to use these in addition to or in place of those given above. At the end of the day, it is hoped that the shape of the graph in a given case might be 'guessed' from the appearance of the equation.

Examples for section 4.1

Discuss the shapes of the graphs in the following cases:

1 A graph of velocity v against time t where the following equation fits the motion:

$$v = 10\,t.$$

2 A graph of distance s against time t where the following equation fits the motion:

$$s = 5\,t^2.$$

3 A graph of velocity v against distance s where the following equation fits the motion:

$$v^2 = 20\,s$$
$$\text{or } v^2 = 20\,s + 10.$$

4 A graph of force F against distance r where the following equation applies:

$$F = 5/r^2.$$

5 A graph of potential V against distance r where the following equation applies:

$$V = 10/r.$$

4.2 Linear graphs from non-linear equations

Section 4.2 gives a simple illustration of another use to which graphs can be put. Theories often lead to equations between quantities and an experiment to check the predicted variation will show whether the theory has failed or not.

The aim of many experiments is to find out how one quantity changes when some other quantity is changed. A graph is a convenient way of displaying the pattern of the measurements, and should that graph be a straight line, it is possible to find an equation relating the values of the dependent variable to those of the independent variable. The question that now arises is: 'If the graph turns out to be a curve, is there any way in which the equation relating the variables can be found?

The class might consider the pattern of values for the distance an accelerating trolley has moved from its rest position after various times:

Time t/s	0.8	1.3	1.9	2.3	2.9
Distance s/m	0.141	0.372	0.794	1.113	1.850

s increases much more rapidly than t and a graph of the data has a shape much like that of $y = kx^2$. In any case, $s \propto t^2$ is expected if the trolley has uniform acceleration, but it is difficult to say with certainty that the s–t graph is of the $s = kt^2$ shape. Had it been a straight line, it would have been much easier. Can the $s = kt^2$ prediction be checked by plotting the figures to obtain a straight line graph? If the s values are plotted as y co-ordinates and t^2 as x co-ordinates and if $s = kt^2$, the equation of the graph would be $y = kx$, which is a straight line through the origin. For this test, the numbers to be plotted are:

$y = s$/m	0.141	0.372	0.794	1.113	1.850
$x = t^2/\mathrm{s}^2$	0.64	1.69	3.61	5.29	8.41

These numbers do give a straight line through the origin and confirm that, in this case, $s = kt^2$, with k equal to 0.22 m s^{-2}. It should be pointed out that plotting the data in this way has enabled the prediction ($s = kt^2$) to be checked and has also resulted in finding the value of k.

Students might now discuss how they would deal with data to see if these fitted the predicted equations in example 1, which follows. In each case, they should consider what information could be gained from a measurement of the gradient and the intercept of the linear graph. Parts **e** and **f** might be attempted if the other parts prove easy, and some students might be helped if the equation is rearranged into the form $y = mx + c$.

For example $F = \dfrac{k}{(r-d)^2}$ rearranges to $r = \sqrt{k}\dfrac{1}{\sqrt{F}} + d$

which can be compared with $y = m \ x + c$.

Examples for section 4.2

1 How would you check to see if experimental data fit the following equations?

a $V = C/r$ where C is a constant.
b $F = k/r^2$ where k is a constant.
c $E = \frac{1}{2}mv^2$ where m is a constant.
d $F = \dfrac{G\,m_1 m_2}{r^2}$

1 when F and m_1 are the only variables,
2 when F and r are the only variables,
3 when m_1 and r are the only variables.

e $V = \dfrac{C}{(r+k)}$ where C and k are constants. k might be an unknown constant error in measuring the distance r of example **a**.

f $F = \dfrac{k}{(r-d)^2}$ where k and d are constants. Again, d might represent a systematic error in measuring r.

[*Possible answers*

	Plot as *y*	Plot as *x*	Gradient gives	Intercept gives
a	V	$1/r$	C	Must go through (0, 0)
b	F	$1/r^2$	k	Must go through (0, 0)
c	E	v^2	$\frac{1}{2}m$	Must go through (0, 0)
d*1*	F	m_1	Gm_2/r^2	Must go through (0, 0)
d2	F	$1/r^2$	Gm_1m_2	Must go through (0, 0)
d3	r^2	m_1	Gm_2/F	Must go through (0, 0)
e	r	$1/V$	C	$-k$
f	r	$1/\sqrt{F}$	$\sqrt{k}$	$+d$

There are other answers, except in the case of **f**.]

2 In checking the predicted equation for the period of a simple pendulum, i.e. $T = k\sqrt{L}$ where T is the period, k a constant, and L the length, an experimenter plots a graph of T against $\sqrt{L}$ and obtains a straight line which does not go through the origin. It cuts the T axis at $+0.1$ second. What do you think this means?

[A systematic error in measuring the period. All the periods are too large by 0.1 second.]

4.3 An unknown index – the use of logarithms

Though the technique is not required in the Physics course, the following might be useful for some practice with logarithms. It deals with the case where variables p and q are related by the equation $p = C\,q^m$, C and m being unknown constants.

As a first step, consider the equation $p = q^m$. Students should be shown that, if logarithms are taken, it becomes $\lg p = m \lg q$ and that this is simply $y = mx$ if $\lg p$ is plotted as the y co-ordinate and $\lg q$ as the x co-ordinate, the gradient of the graph being the unknown index m. Now they should be able to cope with $p = C\,q^m$ for which the $\lg p$ against $\lg q$ graph has a gradient of m and an intercept of $\lg C$. In the examples that follow, the numbers have been chosen to avoid negative characteristics.

Examples for section 4.3

1 Show that the following values of p and q are related by $p = C\,q^m$ and find values for C and m.

p	8.70	12.04	15.16	18.12
q	2.00	3.00	4.00	5.00

[$p = 5q^{0.8}$.]

2 For certain conditions (when heat cannot flow into or out of a gas), the values of the pressure P and the volume V are related by an equation of the type $PV^\gamma = k$ where γ and k are constants. Use the following data to find the value of γ for air.

P/N m^{-2}	1.0×10^5	7.5×10^4	5.0×10^4	2.5×10^4
V/m^3	1.34	1.64	2.19	3.60

[$\gamma = 1.4$.]

4.4 Growth and decay functions

Growth and decay functions are treated in more detail in Section 8, 'Exponential variations'. However, they first appear in the Physics course in Unit 2, and the opportunity is taken here to give students a first look at this kind of pattern.

Considering a simple growth function provides a good introduction. Living matter grows because cells split into two. Suppose that, in a given case, this happens once every minute and that there are 1000 cells to begin with. The class could be asked to produce a table showing how the number of cells changes with time.

Time in minutes t	0	1	2	3	4	5
Number of cells in thousands n	1	2	4	8	16	32

Table 5

The lower numbers in table 5 are the powers of 2 from 2^0 to 2^5. What is the equation which represents the number pattern in this table? Discussion should produce the answer $n = 2^t$. It should be clearly understood why the number 2 appears in the expression, and discussion should extend to the growth functions $n = 3^t$ and $n = 2^{t/2}$. In the latter case, some may prefer to ask what equation would represent a growth process which caused a doubling every two minutes. Graphs of $y = 3^x$, $y = 2^x$, $y = 1^x$ should be plotted on the same set of axes – as in figure 18.

A decay process can be represented by a situation in which half the number of cells alive die in 1 minute. The equation then becomes $n = (\frac{1}{2})^t$ which can be written $n = 1/2^t = 2^{-t}$. A graph of $y = 2^{-x}$ can be plotted on the same axes as those above.

Those who did section 1.2 will have seen a similar growth graph already. There a graph of $y = 10^x$ was plotted and x was said to be the logarithm to base 10 of y. In these graphs, x is the logarithm to a different base of y; e.g. if $y = 3^x$, then $x = \log_3 y$.

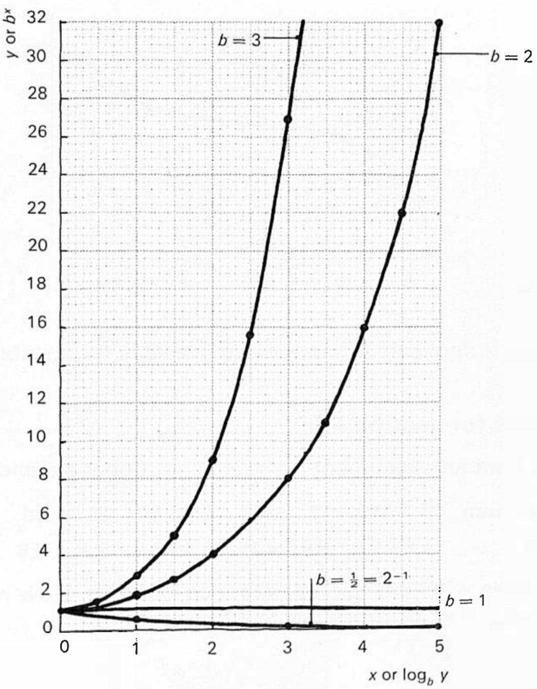


Figure 18

4.5 Area 'under' the graph again

It would be wrong to give the impression that all variations can be expressed by simple algebraic formulae. In a real situation with a wide range of variables, the opposite is the case. The extension of a wire is not proportional to the force producing it if the force becomes too large – the wire breaks! A car does not have a constant acceleration when we consider large times! But a graph of the variation is still useful. Not only does it give a picture of the pattern of the values but it can be used to find the range of the independent variable over which some simple equation describes the variation, and if there is anything significant about its steepness or about area, that can be determined also.

Another look at areas 'under' a graph would be useful here. The class should be given some examples of non-linear variations and asked to find the area under the the graph between certain limits. The method of strip division should be used, i.e. ordinates should be drawn at suitable regular intervals, and a horizontal line drawn at the top of each strip in a position such that the graph passes through the middle of it (figure 19). Pupils who are worried about the inaccuracies of this method could be persuaded to draw the ordinates at a smaller separation and try again. Reminders to measure the height and width of each strip from the scales marked on the axes will be necessary and the units in which the 'area' is measured may need clarification.

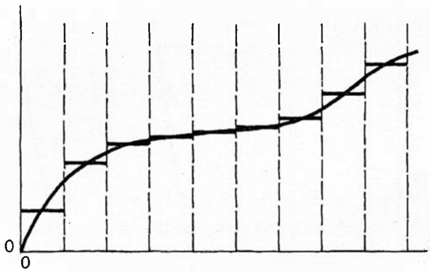


Figure 19

It would save time if duplicated graphs were available for distribution to the class.

Examples for section 4.5

1 Results from an experiment on stretching rubber are shown below:

Extension/mm	5	10	15	20	25	30	35	40	45	50
Force/*N*	1.9	2.9	3.6	4.15	4.6	4.9	5.25	5.5	5.7	5.9

Plot a graph of these with extension as the x co-ordinate. How much energy is transformed in producing an extension of 50 mm?

[0.21 J.]

2 A car makes a journey lasting 6 minutes during which its speed v in km h^{-1} can be represented by the equation

$0.1\ v = 6t - t^2$, t being the time in minutes.

Plot a graph to show how its speed changes with time and estimate how far it travelled.

[Strips 1 minute wide: 6.1 km; strips $\frac{1}{2}$ minute wide: 6.03 km; accurate result is 6.00 km.]

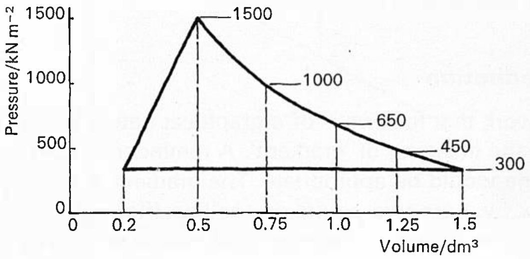


Figure 20

3 Figure 20 shows the variation of pressure with volume for the gas in the cylinder of a certain engine. The useful energy output is represented by the area enclosed within the curve. What is it?

[630 J.]

Section 5

Differentiation

The emphasis in Section 5 is on the graphical interpretation of differentiation. Uniformly accelerated motion is chosen as the situation in terms of which the new ideas are introduced, but it is important that students have practice with examples in which time is not one of the variables and in which the gradient has a physical meaning other than a time rate of change.

5.1 Introduction

Much of the work that follows is of a graphical nature and requires some knowledge of the meaning of 'gradient'. A reminder of how to measure the gradient of a straight line would be appropriate. The gradient is $(y_2-y_1)/x_2-x_1$ where (x_1, y_1) and (x_2, y_2) are two points on the line (figure 21). The symbol Δy is often used to stand for a difference between two values of y, i.e. a y interval, and Δx for

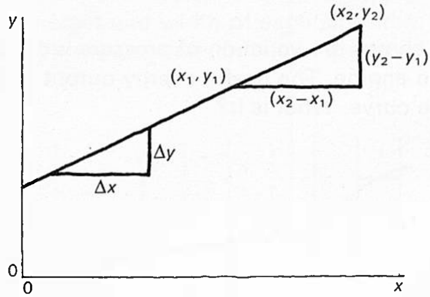


Figure 21

an x interval. These are spoken of as delta y and delta x. So the gradient is $\Delta y/\Delta x$. Ask if it matters where Δx is taken or how big Δx is and bring out the fact that, for a straight line, the gradient is the same everywhere and equal to m in $y = mx+c$. (If this has been forgotten, substitute in $(y_2-y_1)/(x_2-x_2)$.) Ask what the difference is between a graph (figure 22) for which $\Delta y/\Delta x = +5$ and one having $\Delta y/\Delta x = -5$. In the former, as x increases, y increases at 5 times the rate, whereas in the latter as x increases, y decreases at 5 times the rate.

Example for section 5.1

The following are the distance–time data for a particular very small air bubble rising in a liquid, the distance being measured from some arbitrary position.

Time t/s	0	1	2	3	4	5
Distance s/mm	10	20	30	40	50	60

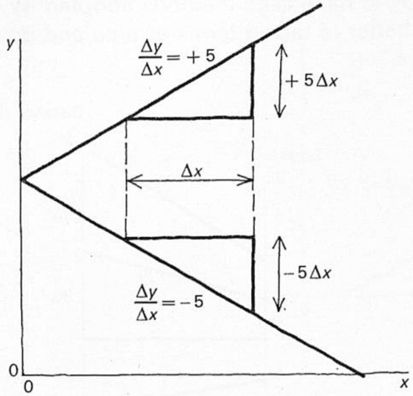


Figure 22

a What is the average speed during the first second?

[10 mm s^{-1}.]

b How does the average speed depend on the time elapsed?

[It is constant.]

c Write down an equation relating the distance with time.

[$s = 10+10t$.]

d What is the gradient of a graph of distance against time and how is it related to the answers to **a** and **b**?

[Gradient = 10 mm s^{-1}; it is constant and equal to the speed.]

5.2 Rate of change and gradient for non-linear variations

An easily visualized example is an accelerating car. All cars carry an instrument which indicates the speed at any instant. The speedometer measures the *instantaneous speed.* When it indicates 40 km h^{-1} (or 25 m.p.h.) it means that if the car exactly maintained the speed of that instant, it would cover a distance of 40 km (or 25 miles) in 1 hour. Data regarding the accelerating car can also be given in the following way:

Distance covered by car, s/m	0	1	4	9	16	25	36	49
Time, t/s	0	1	2	3	4	5	6	7

What can be deduced about the speed from these figures? Can the instantaneous speed at any time be extracted from them? The car is obviously accelerating – during the 1st second it covers 1 m, in the 2nd second 3 m, in the 3rd second 5 m. In the 3rd second, the car has an *average* speed of 5 m s^{-1} – and this was obtained by dividing a distance interval Δs by a time interval Δt.

$$\text{Average speed during interval } \Delta t = \frac{\Delta s}{\Delta t}.$$

For those students who have used methods adopted by recent mathematics projects, it may be better to talk in terms of time and position intervals on mapping diagrams, as follows.

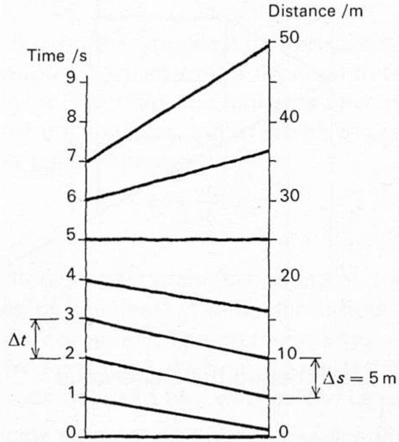


Figure 23

The first mapping diagram (figure 23) shows how time maps into position. The average speed, within the interval, is the distance interval divided by the time interval, i.e. the factor by which the intervals are scaled in mapping from time to position.

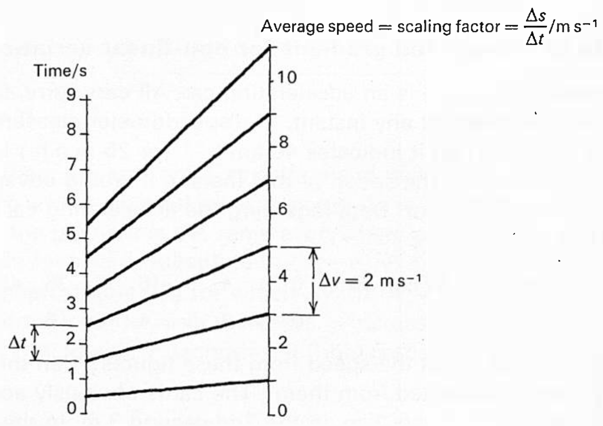
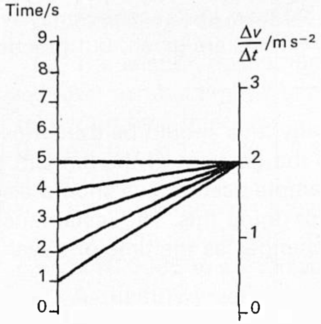
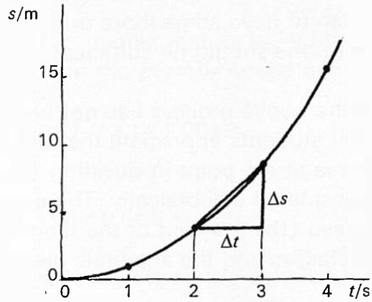


Figure 24

The second mapping diagram (figure 24) shows how this scaling factor, i.e. the average speed, depends on the time. The average speed is proportional to the average time, the acceleration being 2 m s^{-2}. This could be shown by a third mapping diagram (figure 25).





Figures 25 and 26

A graph should be sketched (figure 26) to represent the data given previously and the process of finding the average velocity discussed in graphical terms, to bring out the facts that $\Delta s/\Delta t$ is the gradient of the straight line joining the points on the graph at the beginning and the end of the time interval chosen, and that, if the instantaneous speed at 2 seconds were required, this average would be larger because the car is accelerating. A better estimate of the instantaneous speed at 2 s could be obtained by using a smaller time interval. The class could do some calculations to find the average speeds over time intervals of 0.5 s, 0.1 s, 0.01 s, and 0.001 s and could tabulate the results. The graph cannot be used accurately enough for this purpose, so it will be necessary to tell students that the numbers fit the equation $s = t^2$. The completed table should look like table 6:

$t+\Delta t$	3.0	2.5	2.1	2.01	2.001	since $t = 2$ s
$s+\Delta s$	9.0	6.25	4.41	4.0401	4.004 001	
Δs	5.0	2.25	0.41	0.0401	0.004 001	since $s = 4$ m
Δt	1.0	0.5	0.1	0.01	0.001	
$\Delta s/\Delta t$	5.0	4.5	4.1	4.01	4.001	

Table 6

As Δt gets smaller, the value of $\Delta s/\Delta t$ gets closer and closer to 4.00 m s^{-1}. In fact, the value of $\Delta s/\Delta t$ could be made as close to exactly 4.00 m s^{-1} as was desired by taking a small enough value of Δt. The limiting value of $\Delta s/\Delta t$ as Δt tends to zero is written as $\mathrm{d}s/\mathrm{d}t$ and is called the first differential coefficient or the derivative of s with respect to t.

That is, $\dfrac{\mathrm{d}s}{\mathrm{d}t} = \lim_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t}$.

It should be noted that $\mathrm{d}s/\mathrm{d}t$ and $\Delta s/\Delta t$ are not the same thing but that $\Delta s/\Delta t$ gets progressively closer in value to $\mathrm{d}s/\mathrm{d}t$ as Δt becomes smaller. $\mathrm{d}s/\mathrm{d}t$ is the instantaneous speed (whose value has been found at 2 s).

To most students, the idea of a limiting value will be new and teachers may wish them to have some more practice. Two sample questions are given, but practice with one should be sufficient.

If the above process has not been shown graphically, this should be done now so that students appreciate that the derivative is also the gradient of the tangent to the curve at the point in question (figure 27). The example used above should also be considered algebraically. There is a good reason for doing this. The instantaneous speed (the gradient of the tangent to the curve) changes as the time of travel increases and the algebraic method reveals how.

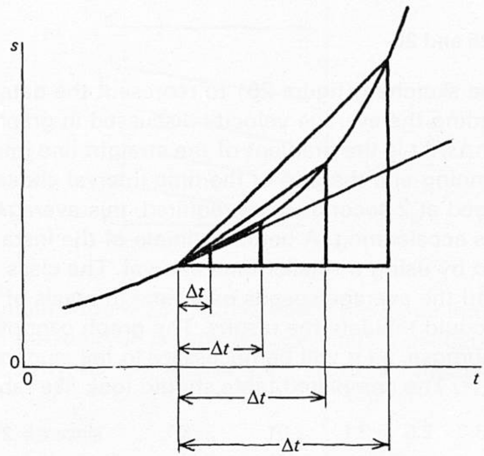


Figure 27

If $s = t^2$, then at time $t+\Delta t$, the distance travelled is $s+\Delta s$.

$\therefore\ s+\Delta s = (t+\Delta t)^2 = t^2+2t\,\Delta t+(\Delta t)^2$.

But $s = t^2$, so we have $\Delta s = 2t\,\Delta t+(\Delta t)^2$ and, dividing throughout by Δt,

$\dfrac{\Delta s}{\Delta t} = 2t+\Delta t.$ (This expression could be checked from the table.)

If Δt gets smaller and smaller, $\Delta s/\Delta t$ will get closer and closer to $2t$, and the limit, when $\Delta t \to 0$, will be $2t$. That is,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 2t.$$

This is the instantaneous speed or rate of change of distance. For a graph of any quantity z, the derivative $\mathrm{d}z/\mathrm{d}t$ measures the instantaneous rate of change of z. If $\mathrm{d}z/\mathrm{d}t = +7$, then, at the time for which the value is quoted, z is increasing at the rate of 7 units s^{-1} whereas if $\mathrm{d}z/\mathrm{d}t = -7$, z is decreasing at the rate of 7 units s^{-1}

There is an opportunity here for a quick revision of O-level kinematics, perhaps in the form of a structured question for homework.

Examples for section 5.2

1 The relation describing how the distance s in metres that an object has moved after various times t in seconds is $s = 5t^2$. Find the average speed during the following time intervals:
a 1.0 s to 2.0 s **b** 1.0 s to 1.1 s **c** 1.0 s to 1.01 s **d** 1.0 s to 1.001 s **e** 1.0 s to 1.0001 s.
What is the instantaneous speed at t = 1.0 s?

[**a** 15 m s^{-1}; **b** 10.5 m s^{-1}; **c** 10.05 m s^{-1}; **d** 10.005 m s^{-1}; **e** 10.0005 m s^{-1}. Exactly 10 m s^{-1}.]

2 A quantity y varies with another quantity x according to the equation $y = \sqrt{x}$. Find the value of $\Delta y/\Delta x$ for the following ranges of x:
a $x = 1.0$ to $x = 2.0$ **b** $x = 1.0$ to $x = 1.5$ **c** $x = 1.0$ to $x = 1.25$ **d** $x = 1.0$ to $x = 1.1$ **e** $x = 1.0$ to $x = 1.05$.
What is the limiting value of $\Delta y/\Delta x$ as $\Delta x \to 0$?
Use four-figure tables to find the square roots.

[**a** 0.41; **b** 0.45; **c** 0.47; **d** 0.49; **e** 0.50. Exactly 0.5.]

3 Sketch velocity–time graphs for **a** a stationary object **b** an object moving with steady velocity **c** a uniformly accelerating body **d** a body whose velocity is decreasing non-uniformly. Your answers to **c** and **d** should show velocities changing with time. Acceleration is rate of change of velocity, $\Delta v/\Delta t$, where Δv is the change in velocity occurring in the time interval Δt. The instantaneous value at a given time is the limiting value of $\Delta v/\Delta t$ as $\Delta t \to 0$, i.e. the gradient of the tangent to the v–t graph at that time, and is written as $\mathrm{d}v/\mathrm{d}t$.
If a body is moving so that the distance travelled, s, is given by $s = kt^2$ where k is a constant, how is the instantaneous velocity related to the time?

[$v = \mathrm{d}s/\mathrm{d}t = 2kt$.]

By taking a time interval Δt in which the velocity increases by Δv, find the value of $\Delta v/\Delta t$.

[$\Delta v/\Delta t = 2k$.]

What is the value of $\mathrm{d}v/\mathrm{d}t$?

[$2k$.]

The acceleration, a, is constant and equal to $2k$ – so it doesn't matter whether Δt is small or not.
Rewrite your equations for s and v using a instead of k and sketch graphs to show how s, v, and a change with time.

[$s = \frac{1}{2}at^2$, $v = at$.]

4 If $s = ut+\frac{1}{2}at^2$, find out how the velocity and the acceleration change with time.

[$v = u+at$, acceleration $= a =$ constant.]

5 Find a formula for ds/dt if $s = t^3$.

[$3t^2$.]

6 Find a formula for the acceleration if $v = 3t^2$.

[$6t$.]

5.3 Differentiation of x^n and rates of change

Up to this point, the development has concentrated on the concepts of distance and velocity and their rates of change. This was partly because that kind of situation was likely to be more familiar and more easily visualized, and partly to avoid introducing other ideas which might have confused the issues. Now students' experience should be widened to cases where the derivative is not a time rate of change, but a rate of change of one quantity with respect to some other quantity which is not time. In doing this, the emphasis should be on the graphical interpretation throughout. As an example, the total power, P, radiated by a lamp filament depends on its absolute temperature, T, approximately as T^4. $\Delta P/\Delta T$ means the average increase in power radiated per unit rise of temperature in the range selected. As ΔT tends to zero, $\Delta P/\Delta T$ becomes the gradient of the tangent to the curve of P against T (figure 28) at a particular temperature, i.e. the 'instantaneous' rate of increase of power with temperature rise. Using a numerical value for dP/dT (e.g. 0.1 W K^{-1} at 2000 K) may help students to get hold of the idea.

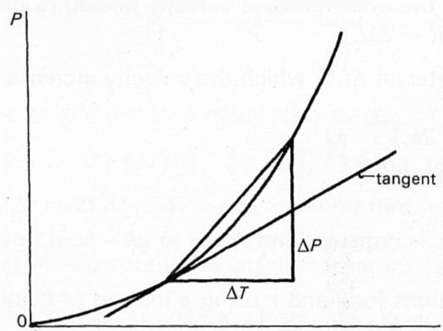


Figure 28

There are cases where the gradient of the tangent to a curve has greater significance. If the energy stored in a spring is E when its extension is x, increasing the extension to $x+\Delta x$ will be accompanied by an increase of energy to $E+\Delta E$. Since the energy transformed equals *force* $\times$ *distance moved* (in the direction of the force), then the average force over the range used is $\Delta E/\Delta x$ for elastic behaviour. The magnitude of the force at a particular value of x is dE/dx.

Other cases will, no doubt, come to mind – electric field as a potential gradient, the change of resistance with temperature, the e.m.f. of a thermocouple as a function of temperature, changes of volume accompanying changes of pressure for a gas. A sufficient number should be discussed for the class to appreciate the importance of derivatives.

Frequently, the variation between two quantities, y and x, involves a relationship like $y = kx^n$ where k and n are constants. How is dy/dx found in that case? The method is the same – consider a change of Δx in x producing a change of Δy in y.

Then $\quad (y+\Delta y) = k(x+\Delta x)^n.$

This can be written as $y+\Delta y = kx^n\left(1+\dfrac{\Delta x}{x}\right)^n$

It is not intended that a treatment of the binomial theorem should follow. For the weaker mathematicians, a simple numerical treatment plus a guess will be enough. Students will know that, at the end, they will imagine Δx becoming very small indeed so that $\Delta x/x$ is a very small number. The problem is:
what is the value of $(1+\text{a very small number})^n$?
Try various values of n for $\Delta x/x = 0.01$.

$$(1+0.01)^2 = (1+0.01)(1+0.01) \approx 1.020$$
$$(1+0.01)^3 = (1+0.01)^2(1+0.01) \approx 1.020+0.010 = 1.030$$
$$(1+0.01)^4 = (1+0.01)^3(1+0.01) \approx 1.030+0.010 = 1.040$$

Each time the result is approximately $1+0.01n$ and the approximation becomes better as $\Delta x/x$ becomes smaller. In general, if $\Delta x/x$ is very small,

$$\left(1+\frac{\Delta x}{x}\right)^n \approx \left(1+n\frac{\Delta x}{x}\right),$$

the error becoming smaller as $\Delta x/x$ becomes smaller.

The expansion of $\left(1+\frac{\Delta x}{x}\right)^n$ when $\frac{\Delta x}{x}$ is very small

The following, more algebraic, argument may suit some students.

With $n = 2$, $\left(1+\frac{\Delta x}{x}\right)^2 = 1\left(1+\frac{\Delta x}{x}\right)+\frac{\Delta x}{x}\left(1+\frac{\Delta x}{x}\right) = 1+\frac{2\Delta x}{x}+\left(\frac{\Delta x}{x}\right)^2$

so we can write $\left(1+\frac{\Delta x}{x}\right)^2 = 1+\frac{2\Delta x}{x}+$ a term involving the square of $\frac{\Delta x}{x}$.

When $n = 3$, $\left(1+\frac{\Delta x}{x}\right)^3 = \left(1+\frac{\Delta x}{x}\right)\left(1+\frac{\Delta x}{x}\right)^2$

$$= \left(1+\frac{\Delta x}{x}\right)^2+\frac{\Delta x}{x}\left(1+\frac{\Delta x}{x}\right)^2$$

$$= 1+\frac{2\Delta x}{x}+\frac{\Delta x}{x}+\text{terms involving the square and cube of }\frac{\Delta x}{x}$$

$$= 1+\frac{3\Delta x}{x}+\text{ terms involving the square and cube of }\frac{\Delta x}{x}.$$

When $n = 4$, $\left(1+\frac{\Delta x}{x}\right)^4 = \left(1+\frac{\Delta x}{x}\right)\left(1+\frac{\Delta x}{x}\right)^3$

$$= 1+\frac{4\Delta x}{x}+\text{terms involving }\left(\frac{\Delta x}{x}\right)^2, \left(\frac{\Delta x}{x}\right)^3, \left(\frac{\Delta x}{x}\right)^4$$

Each time n is increased by 1, the number of terms $\Delta x/x$ increases by 1. This seems to indicate that:

$$\left(1+\frac{\Delta x}{x}\right)^n = 1+n\frac{\Delta x}{x}+\text{terms involving }\frac{\Delta x}{x}\text{ to higher powers than 1.}$$

Going back to $y+\Delta y = kx^n\left(1+\frac{\Delta x}{x}\right)^n$, this now becomes

$$y+\Delta y = kx^n\left(1+n\frac{\Delta x}{x}+\text{terms depending on }\frac{\Delta x}{x}\text{ to higher powers than 1}\right)$$

$$= kx^n+nkx^{n-1}\,\Delta x+\text{the other terms involving }\Delta x\text{ to higher powers than 1.}$$

But $y = kx^n$, and consequently

$\Delta y = nkx^{n-1}\,\Delta x+$terms involving Δx to higher powers than 1.

Teachers using the simpler approach could substitute words like 'a correction much smaller than $\Delta x/x$' for the expression at the end of these equations.

Thus $\Delta y/\Delta x = nkx^{n-1}$ +terms involving Δx.

In the limit when Δx tends to zero, $\Delta y/\Delta x$ tends to nkx^{n-1}.

$\therefore$ if $y = kx^n$, $\boxed{\frac{dy}{dx} = nkx^{n-1}}$

Though it may not occur to students, it has only been shown that this is the result if n is a positive integer. Whether it applies or not when n is a fraction or is negative depends on whether the expression obtained for $(1+\Delta x/x)^n$ remains true. At the least, the class should be told that it does, but no more should be done than to show by simple numerical exercises that calculations come out correctly when n is a fraction or negative, e.g. if $\Delta x/x = 0.01$ and $n = -1$,

$$\left(1+\frac{\Delta x}{x}\right)^n = (1.01)^{-1} \approx 0.9901 \text{ from reciprocal tables,}$$

$$\left(1+\frac{\Delta x}{x}\right)^n \approx \left(1+n\frac{\Delta x}{x}\right) = 0.99,$$

and if $\dfrac{\Delta x}{x} = 0.01$ and $n = \frac{1}{2}$,

$$\left(1+\frac{\Delta x}{x}\right)^n = (1.01)^{\frac{1}{2}} \approx 1.005 \text{ from square root tables,}$$

$$\left(1+\frac{\Delta x}{x}\right)^n \approx \left(1+n\frac{\Delta x}{x}\right) = 1.005.$$

The rule for finding the derivative seems to be valid for all values of n.

The proof that $dy/dx = nkx^{n-1}$ is not to be learned. It is included so that students may see that it can be done and is not prohibitively difficult. They should be encouraged to go through it, but it is the result that is important. Practice questions in quantity will be needed so students may gain familiarity. They should be told that the process of finding dy/dx is called differentiation (because it deals with small differences) and that the derivative dy/dx is sometimes called the differential coefficient of y with respect to x.

Examples for section 5.3

1 The charge Q stored in a battery varies with time t as shown in figure 29. Sketch a graph to show how the current delivered by the battery varies with time. What is happening over the period AB?

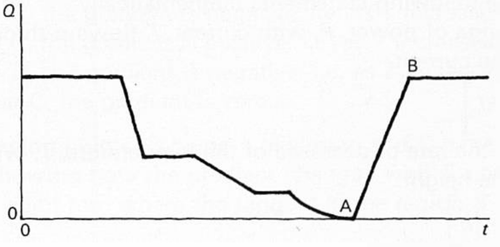


Figure 29

2 The world population at various periods in history has been estimated as:

Date	1650	1750	1800	1850	1900	1950	1960
Population in millions	500	700	900	1100	1600	2500	3000

Sketch a graph showing how the rate of increase of population has changed between 1650 and 1960. What would you estimate the population will be in 1980?

3 A sodium chloride crystal is thought to be made up of regularly arranged charged ions, Na^+ and Cl^-. The electrical tear-apart energy varies with separation as follows:

Separation $r/10^{-10}$ m	2.0	2.8	3.0	3.5	4.0
Energy $E/10^{-19}$ J	20.0	14.3	13.3	11.4	10.0

Draw a graph of E against r.
The normal distance between ions is 2.8×10^{-10} m and the size of the electrical force between them is $\mathrm{d}E/\mathrm{d}r$. Use your graph to find the value of this force at the normal distance of separation.

[5.1×10^{-9} N.]

4 The resistance of a semiconducting material varies with temperature as shown in figure 30. Sketch a graph to show how the rate of change of resistance with temperature varies at different temperatures.

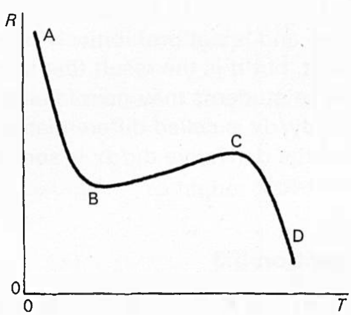


Figure 30

5 Write the following statements mathematically:

a The rate of change of power, P, with current, I, flowing through a resistor is proportional to the current.

$$\left[\frac{\mathrm{d}P}{\mathrm{d}I} = kI.\right]$$

b Near the Earth, the rate of decrease of the temperature, T, with height, h, is proportional to the height.

$$\left[\frac{dT}{\mathrm{d}h} = -kh.\right]$$

c The rate of change of volume, V, of a sphere with radius r, is equal to its surface area. Can you see why?

$$\left[\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2.\right]$$

6 What are the approximate values of the following?
a $(1+0.01)^5$ **b** $(1.005)^{10}$ **c** $(1-0.03)^2$ **d** $(0.995)^3$ **e** $\sqrt{1.004}$ **f** 1.002×1.003
g $1/\sqrt[3]{0.997^2}$ (first write this in the form 0.997^n).

[1.05, 1.05, 0.94, 0.985, 1.002, 1.005, 1.002.]

7 Write down the derivative of y with respect to x in the following cases:
a $y = 3x^2$ **b** $y = 4x^7$ **c** $y = 2x^{10}$ **d** $y = 3x^{-2}$ **e** $y = 4/x^5$ **f** $y = 8/x$ **g** $y = \sqrt{x}$
h $y = 3x^{\frac{1}{3}}$.

[$6x$, $28x^6$, $20x^9$, $-6x^{-3}$, $-20x^{-6}$, $-8x^{-2}$, $\frac{1}{2}x^{-\frac{1}{2}}$, $x^{-\frac{2}{3}}$.]

8 Find the derivative in the following cases:
a $s = ut+\frac{1}{2}at^2$ where u and a are constants
b $y = mx+c$ where m and c are constants.
What can you say about the derivative of a constant?
c $y = 1+2x+3x^2+4x^3$.

[$ds/dt = u+at$; $dy/dx = m$; derivative is 0; $dy/dx = 2+6x+12x^2$.]

9 The power radiated by a hot filament, P, is proportional to the fourth power of the absolute temperature, T. What is the rate of change of radiated power with temperature? What is its value at 2000 K if the power radiated at 2000 K is 100 W?

[$4kT^3 = dP/dT$; 0.2 W K^{-1}.]

5.4 Turning points

This topic serves to consolidate some of the earlier work and to bring attention back to derivatives as gradients of graphs. Some use will be made of turning-points in the work on ionic crystals (Unit 3, *Field and potential*). Indeed, the chapter entitled 'Ionic crystals' in Unit 3 *Students' book* provides a lot of material for the class to work at – and it may be that teachers would prefer to use some of the mathematics time to allow another look at that part of the Physics course. If Unit 3 has not been completed, this topic might be included, as below, in readiness.

The variation shown in figure 30 is that of the resistance of a piece of doped semiconductor over a wide range of temperature. The variation of the gradient of the tangent to the curve, dR/dT, should be discussed, when the following points should emerge:

from A to B, the gradient is negative, i.e. as T increases, R decreases,
from B to C, the gradient is positive, i.e. as T increases, R increases,
from C to D, the gradient is negative, i.e. as T increases, R decreases,
at B and at C, the gradient is zero.

B and C are the turning points, B being a minimum and C a maximum. Another graph, figure 31, showing how the gradient changes with T, could be sketched. This has a turning point too, where the tangent in the region B–C has its greatest positive gradient.

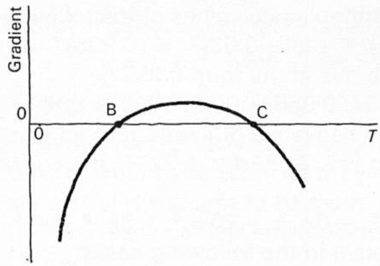


Figure 31

The turning points can be found if the equation describing the data is known, for there the value of the derivative is zero. An examination of the sign of the derivative for x values on either side of the turning point will give the direction of slope of the tangents and indicate whether the turning point is a maximum or a minimum.

Some emphasis needs to be placed on the fact that, for a non-linear graph, the gradient of the tangent to the curve changes from place to place, that is, that dy/dx varies with x. The variation of gradient can be found either by drawing tangents at various x values and measuring their gradients, or, if the curve equation is known, by finding dy/dx. The class should appreciate, qualitatively, that if the graph (figure 32) showing the variation of energy, E, of a pair of ions with their separation, r, is known, a graph showing how the force between them, $-dE/dr$, changes with r can be obtained. Is the reverse true? No; the force $(-dE/dr)$ against r curve only gives the *shape* of the E–r curve at each value of r; that is, it shows what change of E corresponds to a small change of r. It doesn't give the magnitude of E at each point.

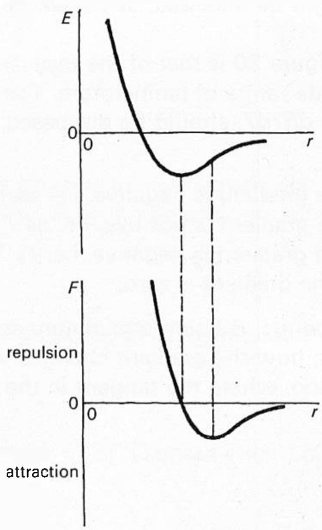
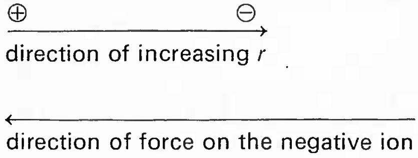


Figure 32

This would be a good opportunity to explain why a negative sign appears in the equation $F = -\mathrm{d}E/\mathrm{d}r$.



If r is increased by moving the negative ion further away against the attractive force, the energy increases too, so that $\mathrm{d}E/\mathrm{d}r$ is a positive quantity.

The force has a direction opposite to that of increasing r and is counted as a negative force. $F = \mathrm{d}E/\mathrm{d}r$ would not do because a negative quantity cannot equal a positive quantity. If the charges were of the same sign, then $\mathrm{d}E/\mathrm{d}r$ would be negative, with the force positive so that F is still equal to $-\mathrm{d}E/\mathrm{d}r$.

Examples for section 5.4

1 A stone is thrown upwards. The distance travelled, s, is given by $s = ut+\frac{1}{2}at^2$ where u is the initial velocity, a is the acceleration, and t is time. In this case $u = 20\ \mathrm{m\ s^{-1}}$ and $a = -10\ \mathrm{m\ s^{-2}}$. Sketch a graph showing how s changes with t.
What is the significance of the turning point?
What is the value of $\mathrm{d}s/\mathrm{d}t$ there?
How long after release is it before the stone reaches its greatest height? What is this height?

[$\mathrm{d}s/\mathrm{d}t = 0$ at turning point; 2 seconds; 20 m.]

2 Find the turning points on a graph of y against x if
a $y = x^2$ **b** $y = x^2-4x+2$ **c** $y = 2+4x-x^2$.
Are these points maxima or minima?

[At $x = 0$, minimum; at $x = 2$, minimum; at $x = 2$, maximum.]

3 There are 2 turning points on the graph of $y = x^3-3x^2$. Where are they? What kind of turning point are they? What are the values of y at the turning points?

[At $x = 0$ and $x = 2$;
at $x = 0$, a maximum with $y = 0$;
at $x = 2$, a minimum, $y = -4$.]

4 The molecule of iodine behaves like two masses joined by a spring, for which the spring stiffness is $3.6\times10^2\ \mathrm{N\ m^{-1}}$. The energy stored in the spring bond when the atoms are distance x apart is given by $E = k(x-L)^2$ where $L = 2.5\times10^{-10}$ m.

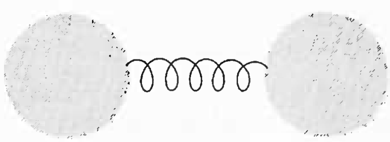


Figure 33

a At what value of x is the force between atoms zero?

[2.5×10^{-10} m.]

b What is the equilibrium separation of the atoms?

[2.5×10^{-10} m.]

c What is the value of k?

[1.8×10^{2} N m^{-1}.]

d The atom is set into oscillation with a total energy of 4.05×10^{-18} J. Draw graphs to show how the potential energy and kinetic energy are related to x.

5.5 The second differential coefficient

In the work on the simple harmonic oscillator in Unit 4, *Waves and oscillations*, second differential coefficients are used. So far in Section 5, there have been one or two cases where, after finding how the derivative of y changes with x, there has been reason to look at the derivative of the derivative (see the examples for section 5.2). In kinematics, if $s = 3t+7t^2$, then the velocity v is given by $v = ds/dt = 3+14t$ and the acceleration is given by $a = dv/dt = 14$. The usual mathematical notation, d^2s/dt^2, should be introduced for dv/dt, the numerals 2 indicating that the process of differentiation with respect to t has been carried out twice. Warnings will be necessary. This is not a fraction; in ds/dt one cannot cancel d and get s/t. It is a piece of mathematical shorthand standing for 'the derivative of s with respect to t was found and then the derivative of that derivative'. It would probably be wise to leave it at that, but some students may be able enough to see the sense in the notation d^2s/dt^2, as follows.

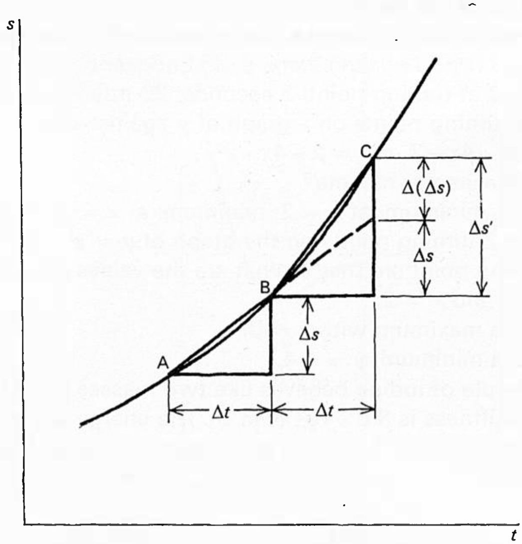


Figure 34

Consider a graph of s against t (figure 34). To find the instantaneous acceleration, the velocity at two times separated by an interval Δt must first be found.
The average velocity over time Δt at A is $\Delta s/\Delta t$ and that at B is $\Delta s'/\Delta t$.
The change in average velocity is

$$\frac{\Delta s'}{\Delta t}-\frac{\Delta s}{\Delta t}.$$

The average acceleration is

$$\frac{1}{\Delta t}\left(\frac{\Delta s'}{\Delta t}-\frac{\Delta s}{\Delta t}\right)$$

which equals $\dfrac{\Delta s'-\Delta s}{(\Delta t)^2}$.
$\Delta s'-\Delta s$ is the change of Δs, or $\Delta(\Delta s)$.
The acceleration is the limiting value of $\Delta(\Delta s)/(\Delta t)^2$ as $\Delta t \to 0$ which is written as d^2s/dt^2.

For many, the difficulties with notation would take too long to overcome.

Example for section 5.5

Find the second differential coefficient of y with respect to x in the following cases:
a $y = 3x^2$ **b** $y = 4x^7$ **c** $y = 2x^{10}$ **d** $y = 3x^{-2}$ **e** $y = 4/x^5$
f $y = 8/x$ **g** $y = \sqrt{x}$ **h** $y = 3x^{\frac{1}{3}}$.
[**a** 6; **b** $168x^5$; **c** $180x^8$; **d** $18x^{-4}$; **e** $120x^{-7}$ **f** $16x^{-3}$; **g** $-\frac{1}{4}x^{-3/2}$; **h** $-\frac{2}{3}x^{-5/3}$.]

5.6 Small changes and 'errors'

This section could be delayed until later, though it would be a pity to omit it altogether. The questions which it attempts to answer should arise during work in the laboratory and it may be better to answer those questions in the Physics course as they arise.

The ideas of the previous sections can be put to an immediate use. There are occasions when the effect on one quantity of a small change in some other quantity needs to be known; or, to put it another way, if x changes and becomes $x+\Delta x$ (Δx small), how does y change? Closely linked to this is the question of 'errors'. In doing experiments, measurements are made and other quantities calculated from them, e.g. to obtain the cross-sectional area of a wire, a diameter measurement is used in a calculation. Inaccuracies of measurement are bound to occur, and so any other quantity calculated from that measurement will be in 'error'. Here the question is: what will be the uncertainty in the result of the calculation if the inaccuracy in the measurement can be estimated?

Often, difficulties experienced in understanding what follows arise from uncertainties about what symbols mean, and though the development here is algebraic, teachers might decide rightly to do the work numerically.

Suppose $y = kx^n$ where x is the variable measured, y being calculated from it by the formula. If there was a small uncertainty Δx in the measurement, then $x+\Delta x$ should have been used in the calculation and not x, and the result would have been $y+\Delta y$ and not y. The uncertainty in y is therefore Δy. A similar argument applies if small changes are being considered; Δy is the change in y resulting from a change of Δx in x. The problem then is to obtain Δy, and section 5.3 showed this to be given by $\Delta y = nkx^{n-1}\,\Delta x$+terms involving higher powers of Δx. If Δx is small compared with x, the terms involving higher powers of Δx can be neglected, so that, to a good approximation,

$$\Delta y \approx nkx^{n-1}\,\Delta x.$$

Point out that this *is* an approximation and that Δx is not, this time, a quantity whose value can be chosen so that it tends to zero. Some may be helped by looking at the above argument graphically (figure 35) and numerical examples should be used to dispel worry about neglected terms.

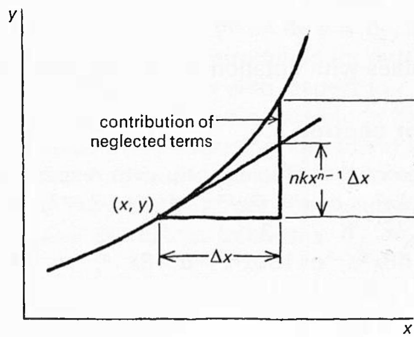


Figure 35

Numerical illustration of effect of neglected terms

As a simple example, let $y = x^2$ and the value of $x = 1$. If the uncertainty in measuring x is 1 per cent, then $\Delta x = 0.01$. For 1 per cent inaccuracy $y = (1.01)^2 = 1.0201$ and the actual 'error' is 0.0201, whereas the 'error' by the approximation is 0.02.

With a 10 per cent error $y = (1.1)^2 = 1.21$, actual 'error' $= 0.21$, estimated as 0.20.
With a 30 per cent error $y = (1.3)^2 = 1.69$, actual 'error' $= 0.69$, estimated as 0.60.
With a 50 per cent error $y = (1.5)^2 = 2.25$, actual 'error' $= 1.25$, estimated as 1.0.

The result of the estimate is good if the percentage error in x is not too large.

Fractional change and combining uncertainties

The fractional uncertainty or change in y is $\Delta y/y \approx (nkx^{n-1}\,\Delta x)/kx^n = n(\Delta x/x)$. $100\,(\Delta y/y)$ is the percentage uncertainty or change in y and the above is usually expressed as:

percentage uncertainty or change in y
≈ n times percentage uncertainty or change in x.

If the class is interested, the topic could be taken further, though it would probably be better if that occurred when the need for it arose in practical work. The next step would be to examine what the uncertainty in y would be if it had to be calculated from the product of two different measurements, e.g. $y = kuv$ where k is a constant. Following an argument similar to that above,

$$y+\Delta y = k(u+\Delta u)\ (v+\Delta v) = kuv+kv\ \Delta u+ku\ \Delta v+k\ \Delta u\ \Delta v$$
$$\Delta y = kv\ \Delta u+ku\ \Delta v+k\ \Delta u\ \Delta v$$

and since $y = kuv$, $\dfrac{\Delta y}{y} = \dfrac{\Delta u}{u}+\dfrac{\Delta v}{v}+\left(\dfrac{\Delta u}{u} \times \dfrac{\Delta v}{v}\right)$.

If both $\Delta u/u$ and $\Delta v/v$ are small, the last term may be neglected in comparison with the others, so that:

percentage uncertainty in y ≈ ***sum*** *of percentage uncertainties in u and v.*

It should be explained why positive signs were given to both Δu and Δv. If it is not known whether the measurements are high or low in calculating the uncertainty in the product, the gloomiest view must be taken to get that *uncertainty*. This point needs careful consideration when y is obtained from the quotient of two numbers, i.e. when $y = ku/v$. If the 'error' in u were such as to make y too large, it is possible that the 'error' in v might also make y too large. Thus:

$$y+\Delta y = k\frac{u+\Delta u}{v-\Delta v} = k\frac{u(1+\Delta u/u)}{v(1-\Delta v/v)} = k\frac{u}{v}\left(1+\frac{\Delta u}{u}\right)\left(1-\frac{\Delta v}{v}\right)^{-1}$$

For small values of $\Delta u/u$ and $\Delta v/v$, this can be approximated as follows:

$$y+\Delta y \approx y\left(1+\frac{\Delta u}{u}\right)\left(1+\frac{\Delta v}{v}\ldots\right) \approx y\left(1+\frac{\Delta u}{u}+\frac{\Delta v}{v}\ldots\right)$$

whence $\dfrac{\Delta y}{y} \approx \dfrac{\Delta u}{u}+\dfrac{\Delta v}{v}$.

Again, *percentage uncertainty in y ≈* ***sum*** *of percentage uncertainties in u and v.*

This can be extended to any number of factors, of course. When a quantity is calculated by multiplying or dividing a number of factors together, the percentage uncertainty in the quantity is approximately equal to the sum of the percentage uncertainties of each of the factors.

Some discussion of random uncertainty and systematic error will no doubt be called for, as well as methods of estimating the random uncertainty in a measurement.

Examples for section 5.6

1 What uncertainty would occur in the calculated value of
a the surface area **b** the volume of a sphere,
if there were a 2 per cent uncertainty in the measurement of the diameter?
[**a** 4 per cent; **b** 6 per cent.]

2 The acceleration due to gravity, g, decreases with an increase in distance R from the centre of the Earth. If $g = k/R^2$, what change in g will result from a 0.01 per cent increase in R?

[0.02 per cent (negative) or g falls by 2 mm s^{-2}.]

3 A satellite is put into a circular orbit round the Earth with a speed v. Due to the variability of rocket fuels, the speed achieved could be 'in error' by 0.01 per cent. What is the percentage variation possible in the orbit time, T? The orbit time T is inversely proportional to the cube of the speed v. What is the percentage 'error' in the orbit radius? The cube of the orbit radius is proportional to the square of the orbit time. What do the negative signs in your answers mean?

[-0.03 per cent; -0.02 per cent.]

4 In Millikan's experiment, what error in the charge of the oil drop (as a percentage) will arise from a 2 per cent uncertainty in the radius of the oil drop?

[3 per cent.]

5 Estimate the possible uncertainty in the value calculated for the Young modulus if there are uncertainties of 0.5 per cent in the value of the stretching force, 0.5 per cent in the measurement of the unstretched length, 1 per cent in the diameter of the wire, and 3 per cent in the extension which the force produces.

[6 per cent.]

6 In an experiment to measure the energy transformed per coulomb of electricity passed, a heater in a metal block was supplied with a current of 1.00 A for 2 minutes, and the temperature rise was measured carefully. After the block had been allowed to cool, it was warmed mechanically by friction. A cord carrying a mass of 10.0 kg was wound round it and the block was turned by a handle so that the mass was held stationary above the floor without any tension in the other end of the cord. It took 120 turns to raise the temperature between the same values as in the case of electrical heating. The diameter of the block was measured and the circumference calculated to be 0.1 m. The value of g was known to be 9.81 m s^{-2} and you can assume the heat losses were identical.
How many joules were transformed per coulomb in the electrical experiment?

[9.81 J C^{-1}.]

Estimate the uncertainty in this result from the information below, and re-state the result as 9 . . . ± . . .

1 The maker of the ammeter only guarantees his meter to be accurate to 2 per cent of full scale deflection (1 A).

2 Each 1 kg mass making up the load was put on a balance which then read 1 kg in each case, but it was found that up to 10 g more could be added before the pointer moved.

3 The diameter of the block was measured with vernier callipers as 31.8 mm (0.0318 m) but a few more measurements ranged from 31.7 to 31.9 mm.

4 The experimenter remembered he had started with the turning handle at the top and finished at the bottom, but couldn't remember whether he called 0 or 1 the first time it reached the bottom.

[$3\frac{1}{2}$ per cent; 9.8 ± 0.4 J C^{-1}.]

Section 6

Sine and cosine graphs

The work of this section is meant to help students to become more familiar with the shapes of the graphs of sines and cosines of angles. The gradients of these graphs are also examined and the circular measure of angles is introduced. Those who have not done section 1.4 may need a reminder about the definitions of the sine ratio and the cosine ratio.

If a teacher wished, Section 7 could be taken before Section 6.

6.1 Introduction

A line rotating about one end can be used to generate the sine and cosine functions. Students should think about a wheel, of 1 m radius, with a number of spokes, and consider the height of the ends of the spokes above the horizontal through the centre. For a spoke making angle θ with the horizontal, this height is $\sin\theta$, and a graph could now be plotted showing how $\sin\theta$ changes for values of θ up to 360° (figure 36).

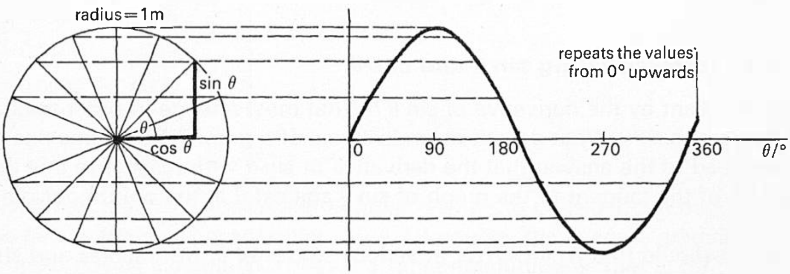


Figure 36

The graph can be checked using a few more values obtained from tables. A teacher could put it to immediate use by asking the class to find the sines of angles larger than 90°. Some students will see relationships between the sines of angles, e.g. $\sin\theta = \sin(180-\theta) = -\sin(180+\theta) = -\sin(360-\theta)$. There is no need to learn these – they follow immediately if one thinks of the wheel and its spokes.

Cos θ is the horizontal distance of the end of a spoke from the centre. A graph of $\cos\theta$ should also be plotted – the shape seems to be the same though the curve is moved 90° to the left compared with the sine graph. Students could cut along the curve with a pair of scissors and superimpose the cosine graph on the sine graph to show that the shapes are identical.

Some teachers may like to do a demonstration using some of the apparatus required for demonstration 78, Nuffield O-level Physics *Guide to experiments V*, to get these ideas over. Use the lamp and the turntable plus sphere and screen. The turntable can be turned through various angles by hand and the projected displacement of the shadow of the sphere measured. The arithmetic will be easier if the maximum displacement from the centre is a convenient number for division.

Examples for section 6.1

1 What are the values of the sines of the following angles?
a 0° **b** 90° **c** 180° **d** 270° **e** 360°.
[0, 1, 0, −1, 0.]

2 What are the cosines of the above angles?
[1, 0, −1, 0, 1.]

3 What is the range of angles, within the range 0° to 360°, for which, respectively, **a** the sine **b** the cosine is negative?
[**a** 180° to 360°; **b** 90° to 270°.]

4 Which angle or angles have a sine of value
a +0.5 **b** −0.5 **c** +0.71 **d** −0.71? (Think of the wheel and its spokes.)
[**a** 30° and 150°; **b** 210° and 330°; **c** 45° and 135°; **d** 225° and 315°.]

5 Repeat example **4** for the cosines. (Think of the wheel and its spokes again.)
[**a** 60° and 300°; **b** 120° and 240°; **c** 45° and 315°; **d** 135° and 225°.]

6 How does a graph of $y = 3 \sin \theta$ differ from a graph of $y = \sin \theta$?

6.2 Differentiating sin *θ* and cos *θ*

What is meant by the derivative of $\sin \theta$? What must change in order to change the value of $\sin \theta$? What does it mean in terms of a graph? Questions like these should lead to the answer that the derivative of $\sin \theta$ with respect to θ is the gradient of the tangent to the graph of $\sin \theta$ against θ at the point considered.

Students should find $\Delta(\sin \theta)/\Delta\theta$ at various values of θ, from tables and should tabulate their calculation as in table 7*a*.

Differentiating sin *θ* numerically

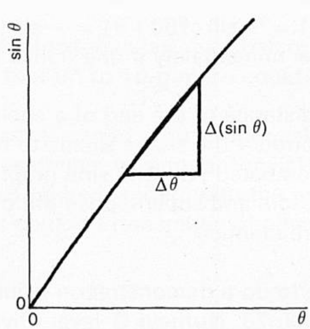


Figure 37

θ	$\theta+\Delta\theta$	$\sin\theta$	$\sin(\theta+\Delta\theta)$	$\Delta(\sin\theta)$	$\Delta\sin\theta/\Delta\theta$
0°	1°	0.0000	0.0175	0.0175	0.0175
15°	16°	0.2588	0.2756	0.0168	0.0168
30°	31°	0.5000	0.5150	0.0150	0.0150
45°	46°	0.7071	0.7193	0.0122	0.0122
60°	61°	0.8660	0.8746	0.0086	0.0086
75°	76°	0.9659	0.9703	0.0044	0.0044
90°	—	—	—	—	0.0000

a

θ	$\cos\theta$	$0.0175\cos\theta$
0°	1.0000	0.0175
15°	0.9659	0.0169
30°	0.8660	0.0152
45°	0.7071	0.0124
60°	0.5000	0.0088
75°	0.2588	0.0045
90°	0.0000	0.0000

b

Table 7

Any who feel that 1° is too big an angular change to consider can be invited to do the calculations using a smaller value. Of course, the average gradient at, say, 130°, need not be calculated. It has the same magnitude as that at 50° but a negative sign. Values of $\Delta(\sin\theta)/\Delta\theta$ can now be plotted over a range from 0° to 360°.

The shape of this graph is suspiciously like a cosine graph, but the largest value is only 0.0175. The class might suggest that the expression describing the graph of $\Delta(\sin\theta) / \Delta\theta$ against θ is $0.0175 \cos\theta$ and they could then check to see if values of $0.0175\cos\theta$ tallied with the values for $\Delta(\sin\theta)/\Delta\theta$ at a few points (table 7*b*). If the suggestion is not forthcoming, some time must be spent discussing how a graph of $k\cos\theta$ differs from a $\cos\theta$ graph.

The agreement is striking and becomes better as $\Delta\theta$ becomes smaller.

In the limit, $\Delta\theta \to 0$.

$$\frac{d(\sin\theta)}{d\theta} = -0.0175\cos\theta \text{ per degree of angle.}$$

It should not be necessary to do the case of the cosine in the same detail; a qualitative examination of a cosine graph shows that the gradient variations are exactly opposite to the variations of a sine graph.

$$\frac{d(\cos\theta)}{d\theta} = -0.0175 \sin\theta \text{ per degree.}$$

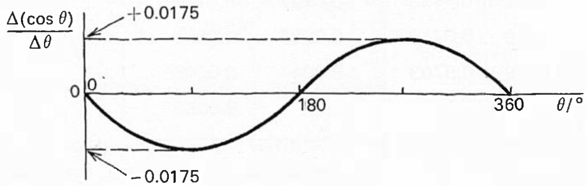


Figure 38

6.3 Circular measure of angles

Two treatments are given below, **A** and **B**. **A** is a simple empirical approach which might suit the less able mathematicians. **B** is more analytical and uses the circle method of section 6.1. Teachers should choose one of these treatments or use their own method according to the ability of the class. The class should not be made to feel that the radian is a mysterious unit of angle but rather that it is a sensible unit because it simplifies equations for the derivatives of $\sin\theta$ and $\cos\theta$.

A The factor 0.0175 comes into the expressions in section 6.2 because the angles were measured in degrees. This should become clear if the meaning of the derivative is considered. For simplicity, consider the gradient of the tangent to a sine curve (figure 39) at the point where $\theta = 0°$, or $360°$ (or $180°$). At that point, the size of $\cos\theta$ is exactly 1 and the value of $d(\sin\theta)/d\theta$ is 0.0175 per degree of angle. What does this mean? If a tangent to the sine curve is drawn at the angle $\theta = 0°$, the straight line has a gradient of 0.0175 per degree of angle, i.e. it rises (or falls) by 0.0175 for every 1° increase of angle. If the unit of angle had been different, the rise (or fall) of the tangent would have been different for unit change of the angle.

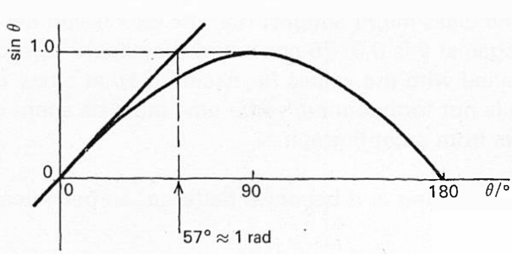


Figure 39

How many degrees must the angle change by for the tangent to rise from 0.0 to 1.0? This can be found by drawing a tangent at 0° on the graph, or by calculation.

It is a change of $1.0/0.0175° = 57.14°$ and if this had been the unit in terms of which angles were measured, the multiplying factor in the equations would have been unity. This is so convenient that the angle (it turns out to be $57.30°$ if calculations are done more accurately) is used as a unit in which other angles can be measured. It is called 1 *radian* and books of tables usually include one for the conversion of degrees into radians. If angles are measured in radians:

$$\frac{\mathrm{d}(\sin\theta)}{\mathrm{d}\theta} = \cos\theta \ \mathrm{rad}^{-1} \quad \text{and} \quad \frac{\mathrm{d}(\cos\theta)}{\mathrm{d}\theta} = -\sin\theta \ \mathrm{rad}^{-1}.$$

There is more significance than there seems to be in this method of measuring angles. A clue can be found by calculating how many radians there are in 360° (6.28) – the result appears to be 2π. So, if an angle of 1 rad were drawn at the centre of a circle (figure 40), AB would fit into the circumference 2π times. Arc AB would be equal in size to the radius, and arc AB/OA = 1. This is how angles are measured in radians. An arc of radius r is drawn across the angle and the arc length s within the angle is measured (figure 41). Then

$$\text{angle } \theta \text{ in radians} = \frac{\text{arc length } s}{\text{arc radius } r}.$$

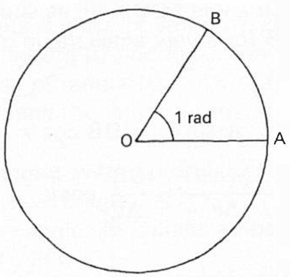


Figure 40

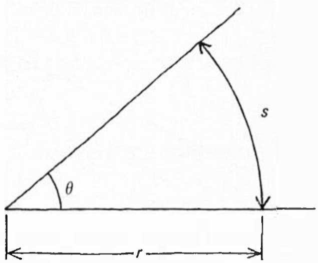


Figure 41

B The reason for the appearance of 0.0175 in these expressions becomes apparent if $d(\sin\theta)/d\theta$ is obtained by the circle method. To achieve this, the angle θ must be changed by $\Delta\theta$, the change of $\sin\theta$ determined, and the limiting value of $\Delta(\sin\theta)/\Delta\theta$ found when $\Delta\theta$ tends to zero. Figure 42 shows this. AB is $\sin\theta$, CD is $\sin(\theta+\Delta\theta)$ and thus the change in $\sin\theta = \Delta(\sin\theta) =$ DE.

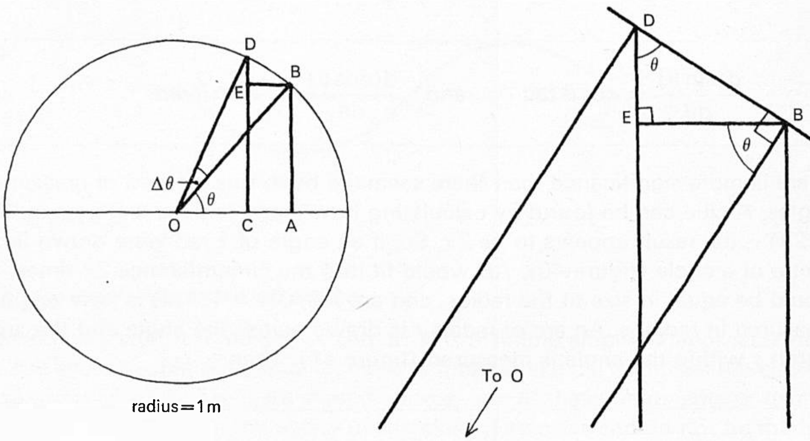


Figure 42

As $\Delta\theta$ becomes smaller and smaller, the figure EDB gets closer and closer to a right-angled triangle in which angle EDB $= \theta$, as shown in the diagram on the right, an enlargement of EDB for a very small value of $\Delta\theta$. From triangle EDB, in the limit when $\Delta\theta$ tends to zero,

$$DE = DB\cos\theta \quad \text{or} \quad \Delta(\sin\theta) = DB\cos\theta.$$

$$\text{Hence} \quad \lim_{\Delta\theta\to 0} \frac{\Delta(\sin\theta)}{\Delta\theta} = \frac{DB}{\Delta\theta}\cos\theta.$$

DB is the same fraction of the circumference as $\Delta\theta$ is of 360° so that $DB/\Delta\theta = 2\pi/360$ (circle radius = 1 m) and $2\pi/360 = 0.0175$. The 0.0175 appears in the equations because the angles were measured in degrees.

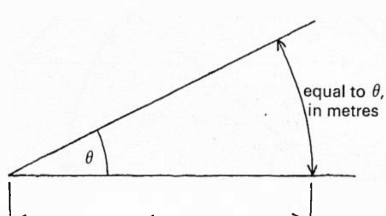


Figure 43

The angle could have been measured in units to make $DB/\Delta\theta = 1$. For angles not small, this would require the arc length on a circle of unit radius to be equal to the angle, so that there are 2π of these units in 360°. This unit is the *radian*; 1 radian is 57.30°. If the radius of the circle is r, then the arc length s equals θr (figure 43). Thus

$$\boxed{\theta \text{ in radians} = \text{arc length } s/\text{arc radius } r}$$

Note also that $\lim \Delta\theta \to 0 \quad \dfrac{\Delta(\cos\theta)}{\Delta\theta} = -\dfrac{DB}{\Delta\theta}\sin\theta$

$$\boxed{\frac{d(\sin\theta)}{d\theta} = \cos\theta \ \text{rad}^{-1} \quad \text{and} \quad \frac{d(\cos\theta)}{d\theta} = -\sin\theta \ \text{rad}^{-1}.}$$

Of course, measuring angles in radians does not change the shape of the graphs. All it amounts to is that, instead of marking the angle scale from 0° to 360°, the scale is marked in radians at the rate of 1 radian = 57.30°.

Practice will be needed to make the radian familiar to the class.

Examples for section 6.3

1 Measure an angle with a piece of string by the following method. Draw an angle of about 120° (don't measure it) and make the lines defining the angle about 12 cm long. Now draw a circle of radius 10 cm with its centre at the apex of the angle. Use the string to determine the length of the arc cut off by the angle. Calculate the angle in radians.
Now measure the angle in degrees with a protractor and use tables to find out how accurate your measurements have been.

2 Express the following angles in radians without using tables.
30°, 60°, 90°, 45°, 180°, 270°, 360°.
[$\pi/6$, $\pi/3$, $\pi/2$, $\pi/4$, π, $3\pi/2$, 2π.]

3 Express the following angles in degrees.
$\pi/4$, $2\pi/5$, $2\pi/3$, 0.925, 0.740, all in radians.
[45°, 72°, 120°, 53°, 42.4°.]

4 What are the values of the sines of the following angles?
1 rad, $\pi/2$ rad, 0.3840 rad.
[0.8415, 1.0000, 0.3746.]

5 Without using tables, convert the following angles in degrees into radians.
40°, 300°, 18°, 720°, 180°.
[$2\pi/9$, $5\pi/3$, $\pi/10$, 4π, π.]

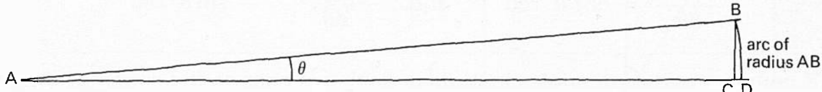
6 A wheel on a ticker tape trolley has a diameter of 40 mm. The trolley rolls through a distance of 1 m. What angle in radians does the wheel turn through?
[50.]

7 A sector of a circle of radius r is of angle θ rad. What is its area?
[$\frac{1}{2}\theta r^2$.]

8 How far round the arc of a circle is it if the arc subtends an angle of 3 radians at the centre and the radius is 60 mm?
[180 mm.]

6.4 Small angles

It should only be necessary to draw up a table (table 8) showing the values of θ in radians, $\sin\theta$, $\cos\theta$, $\tan\theta$ for angles in the range 0° to 10° to make the point that, for these angles, $\sin\theta \approx \theta \approx \tan\theta$ and $\cos\theta \approx 1$, the approximations being the better the smaller the angle. The reason is apparent if a small angled right-angled triangle (figure 44) is drawn. For small θ, BD $\approx$ BC and AC $\approx$ AB, so giving the approximate equalities.



$\sin\theta \approx \frac{BC}{AB}$ $\quad \theta \approx \frac{BD}{AB}$ $\quad \tan\theta \approx \frac{BC}{AC}$ $\quad \cos\theta \approx \frac{AC}{AB}$

Figure 44

θ/°	θ/rad	$\sin\theta$	$\cos\theta$	$\tan\theta$
10	0.1745	0.1736	0.9848	0.1763
5	0.0873	0.0872	0.9962	0.0875
2	0.0349	0.0349	0.9994	0.0349
1	0.0175	0.0175	0.9998	0.0175

Table 8
Values for small angles.

6.5 The derivatives of $A \sin k\theta$ and $A \cos k\theta$

No doubt it will be easier if teachers discuss examples in which A and k have numerical values, and if they deal first with $A \sin\theta$ and then with $\sin k\theta$. The results obtained for sine can be 'carried over' to the cosine case. It may be best simply to offer the examples (page 67) and guess the answer in the general case.

A graphical approach is advised (figure 45). The effect of A in $A\sin\theta$ is to change all values of the y co-ordinate for a definite θ by a factor A. Thus for a given $\Delta\theta$ at a certain value of θ, $\Delta(\sin\theta)$ becomes $A\,\Delta(\sin\theta)$, and the gradient will be A times larger.

$$\frac{d(A\sin\theta)}{d\theta} = A\cos\theta.$$

What about $\sin k\theta$? A graph of $\sin\theta$ against θ is identical with a graph $\sin k\theta$ against $k\theta$. Obtaining a few points for the graphs will convince any who doubt it. However, if the graph is to be of $\sin k\theta$ against θ, then the scale on the axis will need re-numbering if re-drawing the curve is to be avoided – figure 46 shows this.

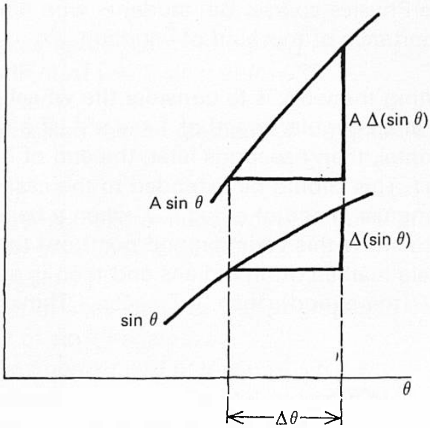


Figure 45

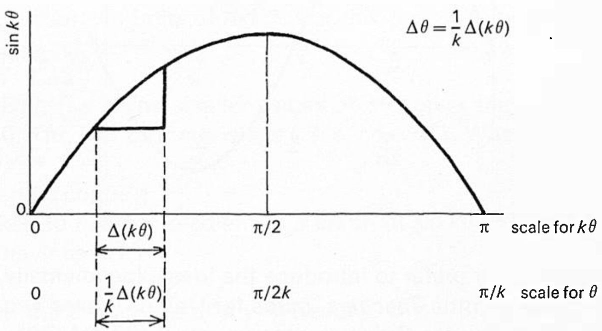


Figure 46

The average gradient of $\sin k\theta$ against θ is

$$\frac{\Delta(\sin k\theta)}{\Delta\theta} = k\frac{\Delta(\sin k\theta)}{\Delta(k\theta)},$$

i.e. k times larger than that of a $\sin k\theta$ against $k\theta$ graph.

$$\therefore \frac{d(\sin k\theta)}{d\theta} = k\cos k\theta.$$

If the effects of both the A and the k are taken into account:

$$\frac{d(A\sin k\theta)}{d\theta} = Ak\cos k\theta, \quad \text{and similarly} \quad \frac{d(A\cos k\theta)}{d\theta} = -Ak\sin k\theta.$$

6.6 Angular rotation and $y = A \sin \omega t$

The aim of this section is to introduce the ideas of angular velocity and variation with time described by equations of the type $y = A \sin \omega t$ and $y = A \cos \omega t$. It is not meant to be a study of systems which are described by such equations; that is done in Unit 4 of the Physics course. But students who have already done Unit 4 will know of the importance of this kind of variation.

A simple way of starting the work is to consider the wheel, thought about in section 6.1, rotating at an angular speed of 1 rad s^{-1}. If a clock is started when a given spoke is horizontal, then t seconds later, the end of it is at a height above the horizontal of $y = \sin t$. This should be extended to the case of a wheel of radius A turning at a steady angular speed of ω rad s^{-1} when y becomes $A \sin \omega t$. Sketch a graph (figure 47) to show this variation and point out the comparison between the x co-ordinate scale marked off in radians and then in seconds. If the time for one cycle of changes is T (the period), then, $\omega T = 2\pi$. ('Think in terms of what the wheel is doing.')

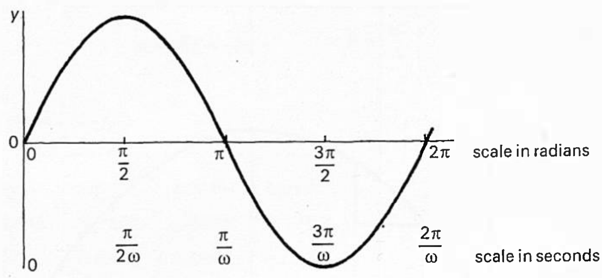


Figure 47

Some teachers may prefer to introduce the idea experimentally, using one of the methods detailed in the *Teachers' guide* for Unit 4, *Waves and oscillations*, or in the Nuffield O-level Physics *Guide to experiments V* (73, 74, 78).

Questions about the speed of the end of a spoke in a vertical direction will raise the problem of finding the derivative of $y = A \sin \omega t$. By comparison with the result obtained in section 6.5,

$$\frac{d}{dt}\left(A \sin \omega t\right) = A\omega \cos \omega t \quad \text{and} \quad \frac{d}{dt}\left(A \cos \omega t\right) = -A\omega \sin \omega t.$$

It may help to consolidate the work of the previous section if students consider the effects on the vertical speed of a point on the wheel as A and ω are changed.

Examples for sections 6.5 and 6.6

1 On the same set of axes, draw or sketch graphs to show how the variation of $\sin\theta$ against θ compares with that of $2\sin\theta$ against θ in the range 0° to 360° Measure or calculate (by considering a value of $\Delta\theta = 1°$) the gradient of each graph at 30°.

[0.015 per degree and 0.030 per degree.]

What would be the gradient of $y = 5\sin\theta$ at $\theta = 30°$?

[0.075 per degree.]

What is the general expression for the gradient of the graph of $\sin\theta$ against θ if θ is in radians?

[$\cos\theta$.]

What do you think the general expressions would be for a graph of $A\sin\theta$ against θ and for a graph of $A\cos\theta$ against θ?

[$A\cos\theta$ and $-A\sin\theta$.]

2 On the same set of axes, draw or sketch graphs to compare the variation of $\sin\theta$ against θ with that of $\sin 2\theta$ against θ.
Measure or calculate the approximate gradient of $\sin\theta$ at $\theta = 60°$ using $\Delta\theta = 1°$.
Measure or calculate the approximate gradient of $\sin 2\theta$ at $\theta = 30°$ using $\Delta\theta = 1°$.
What would be the result of doing this for $\sin 3\theta$ at $\theta = 20°$?

[0.0086 per degree; 0.017 per degree; 0.025 per degree.]

Now obtain these gradients in units of rad^{-1}. For this purpose, they should be multiplied by 57.3 degrees rad^{-1}.

[0.49; 0.97; 1.43.]

More accurate working (i.e. using smaller values of $\Delta\theta$) gives the gradients as 0.50, 1.00, and 1.50. The first of these is $\cos\theta$ (i.e. $\cos 60°$). What are the second and third?

[$2\cos 2\theta$ and $3\cos 3\theta$.]

If you had been asked to find the gradient of a graph of $\sin k\theta$ against θ, what should the answer have been?

[$k\cos k\theta$.]

3 Write down the derivatives of **a** $y = 5\sin 3x$ **b** $y = 3\cos 5x$.
In each case, calculate the gradient of the graph when $x = \pi/3$ rad.

[$dy/dx =$ **a** $15\cos 3x$; **b** $-15\sin 5x$. Gradient = **a** $-15\ \text{rad}^{-1}$;
b $+13\ \text{rad}^{-1}$.]

4 Write down the first five angles at which there are turning-points on a graph of **a** $y = 2\sin x$; **b** $y = 3\cos 6x$.

[**a** $\pi/2, 3\pi/2, 5\pi/2, 7\pi/2, 9\pi/2$; **b** $0, \pi/6, \pi/3, \pi/2, 2\pi/3$.]

5 What is the significance of the constants a and ω in the expression $y = a\sin\omega t$ for the position of an object at various times?

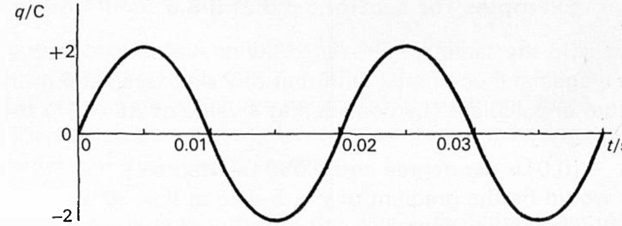


Figure 48

6 The graph (figure 48) shows the variation of the charge on a capacitor with time. The variation can be described by the equation $q = q_0 \sin \omega t$.
What are the values of q_0 and ω?

[$q_0 = 2$ C, $\omega = 100\pi$ rad s^{-1}.]

What, in general, is the current flowing?

[$\omega q_0 \cos \omega t$.]

How large is the current when $t = 1$ s?

[200π A.]

7 Figure 49 is a copy of a ticker-tape trace, obtained by attaching a piece of ticker tape to one end of a pendulum. Plot a graph to show how the displacement from the central position y varies with time t.

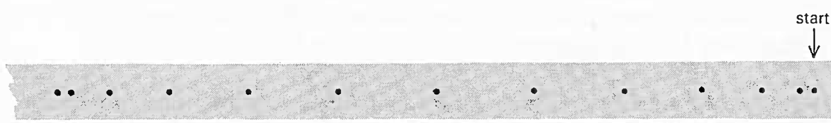


Figure 49

What is the equation describing this motion if the vibrator was vibrating at 50 Hz and only every fifth tick is shown on the above copy?

[$y = 50 \cos (\pi t/1.2)$ mm.]

How does the speed of the pendulum bob change with time?

[$dy/dt = -(50\pi/1.2) \sin (\pi t/1.2)$ mm s^{-1}.]

What is its maximum speed?

[$50\pi/1.2 = 131$ mm s^{-1}.]

How does this check with the average speed at the centre found directly from the tape?

[130 mm s^{-1}.]

8 The motion of an oscillating mass on a spring can be described by:

$$y = 0.1 \sin 10t$$

where y is the displacement in metres from the centre of its motion, and t is the time in seconds.

What is **a** the speed **b** the acceleration of the mass?

[$\cos 10t$; $-10 \sin 10t$.]

What is **a** the speed **b** the acceleration, when $t = 0$ s and when $t = \pi/20$ s?

[Speed: 1 m s^{-1}; 0. Acceleration: 0; -10 m s^{-2}.]

What do you think a negative acceleration means?

Section 7

Integration

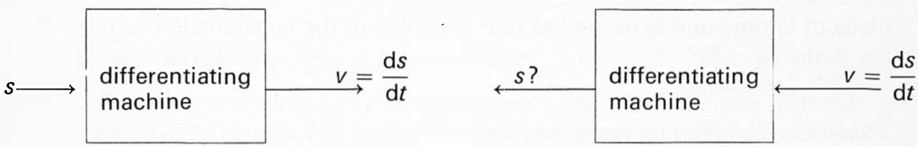
Section 7 deals with integration, first as the reverse of differentiation, then by the numerical method, and finally as the area under a graph. The technique of numerical integration is important as it is the method by which some differential equations are solved in the Physics course. The method is used in Unit 2, *Electricity, electrons, and energy levels*, and then again in Units 3, 4, and 5, *Field and potential*, *Waves and oscillations*, and *Atomic structure*, in the first year's work. Unit 10, *Waves, particles, and atoms*, also uses the ideas. For those who have not yet done Unit 2, this section would form a helpful introduction; for those who have met the ideas already in their physics, it provides the opportunity for a second look, to consolidate and get things straight in their minds. It is worth reminding teachers that the *Students' books* for the Units mentioned provide some examples.

This section could easily be taken immediately after Section 5 if desired.

7.1 Reverse differentiation

The case of motion under constant acceleration is chosen to illustrate the work because of its familiarity, but, if it is felt that the class will find it uninteresting, one of many other examples could be used.

The problem could be stated as follows: 'We know how to find the velocity and acceleration of a body if we are told how the distance it travels, s, is related to time. We have to differentiate s with respect to time to get the speed and then again to get the acceleration. Supposing we knew what the acceleration was, can we get the equations for the velocity and the distance travelled? Can we work the differentiation process in reverse?'



Suppose the acceleration dv/dt were constant and equal to 10 m s^{-2}. This simply means that the gradient of the velocity–time graph is constant and the v–t graph is therefore a straight line. But there are very many different straight lines which can be drawn with the same gradient, each having a different intercept from the others (figure 50). The equation which describes a straight line graph is $y = mx+c$ where m is the gradient and c the intercept.

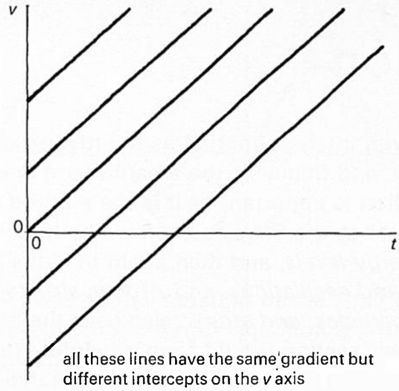


Figure 50

Thus, if all that is known is $dv/dt = 10$, the equations for the possible velocity–time graphs have the form:

$$v = 10t + c.$$

What is the significance of c? It is the velocity when $t = 0$. If this information is given, or indeed, a velocity value at any particular time (i.e. one point on the v–t line), then c can be found. No doubt, the equation will be recognized as $v = u + at$.

What about the distance covered if the body starts with $v = 0$ at $t = 0$ (i.e. $c = 0$)? For these conditions, $v = ds/dt = 10t$. What has to be differentiated to obtain $10t$ as the result? Differentiating $y = kx^n$ gives $dy/dx = nkx^{n-1}$ so $n = 2$ and $k = 5$. As before, any constant could be added to the equation for s, because a constant doesn't vary and has a derivative of zero.

$$s = 5t^2 + constant.$$

Reverse differentiation doesn't give the whole answer – on each occasion one more piece of information is needed to find the value of the constant; if $s = 0$ when $t = 0$, then $s = 5t^2$.

[$s = \frac{1}{2}at^2$.]

Students will need practice examples.

Examples for section 7.1

1 The force F in newtons needed to stretch a certain spring by distance x is found to be equal to $60x$ when x is measured in metres. The energy converted to spring energy, ΔE, in a small extension Δx is given by

$$\Delta E = F\,\Delta x.$$

Write this as a differential equation involving dE/dx and x and then solve it, i.e., find an expression for E which, when differentiated, gives the differential equation.

[$E = 30x^2 + c$.]

How much work must be done to stretch the spring from 10 mm to 60 mm?

[$E = 0.105$ J.]

2 What are the 'reverse differentiation' equations for **a** $dy/dx = 5x$ **b** $dy/dx = 5x+3$?

[$y = (5x^2/2)+c$; $y = (5x^2/2)+3x+c$.]

3 How does the distance s vary with time t for an object accelerating from an initial velocity of u at a uniform acceleration a, i.e. $v = u+at$?

[$s = ut+\frac{1}{2}at^2+c$.]

4 Find equations for curves having the following gradients:

a $dy/dx = 2$.

[$y = 2x+c$.]

b $dy/dx = 2x$.

[$y = x^2+c$.]

c $dy/dx = 2x^2$.

[$y = \frac{2}{3}x^3+c$.]

d $dy/dx = 2x^3$.

[$y = \frac{1}{2}x^4+c$.]

e $dy/d\theta = 10 \cos 5\theta$.

[$y = 2 \sin 5\theta+c$.]

f $dy/d\theta = 10 \sin 5\theta$.

[$y = -2 \cos 5\theta+c$.]

5 A body moves in a straight line with an acceleration equal to $6(1-t)$ in m s^{-2}, where t is the time for which it has been travelling. It starts from rest, and 1 second later, it reaches a point 2 m from where it started. How far will the body be from its starting point after **a** 2 s **b** 3 s?

[$s = 3t^2-t^3$; **a** 4 m; **b** 0 m.]

6 The speed of a body in metres per second at various times is given by $v = 4t^2-3$. Rewrite this as a differential equation involving ds/dt and t. Find the relation between s and t which corresponds with this and then calculate the distance travelled between $t = 2$ s and $t = 5$ s.

[$s = \frac{4}{3}t^3-3t+c$; 147 m.]

7.2 Numerical 'reverse differentiation'

The method of numerical 'reverse differentiation' is taught within the Nuffield Advanced Physics course. The following will serve as an introduction or as a revision for those needing extra practice.

Graphically, the equation $dy/dt = k$ defines a set of lines of gradient k. Given the value of y at any time, the line defined can be obtained by a numerical method.

If Δy and Δt are increments of y and t, $\Delta y/\Delta t$ is the average gradient over the time interval Δt at the place considered, and equal to that of the tangent. Then $dy/dt = \Delta y/\Delta t$. (This is dealt with in section 5.1.)
Suppose that the initial value of $y(t = 0)$ is 2 m and that $dy/dt = 5$ m s^{-1}.
In a time interval of 0.1 s ($\Delta t = 0.1$ s), y increases by $\Delta y = 5\Delta t = 0.5$ m.
Therefore y becomes 2.5 m at a time $t = 0.1$ s.

In the next interval $\Delta t = 0.1$ s, Δy is again 0.5 m, and $y = 3.0$ m at the end of that interval.

It is apparent that this is going to give a straight line (figure 51).

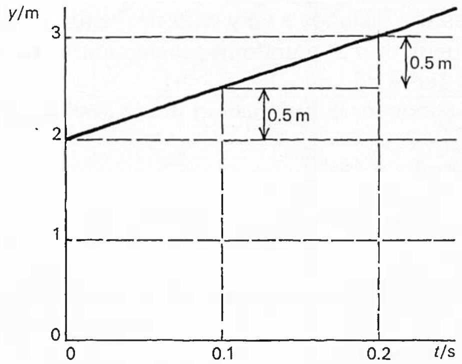


Figure 51

Solution of $dy/dt = 5y$

The method can now be used to solve a more difficult differential equation, e.g. $dy/dt = 5y$.

It might be worth while spending a few moments letting the class guess at a solution. ('No, it isn't $\frac{5}{2}y^2$ because its a y–t graph and we are differentiating with respect to t'; 'it isn't $5yt$ because the y on the right isn't a constant and in any case, you couldn't have $y = 5yt$.') This is a graph whose equation hasn't been differentiated up to now.

As before, the equation is written in terms of small intervals of y and t:

$$\Delta y \approx 5y\,\Delta t.$$

This equation can now be used to build up the y–t curve if a point on the curve is known (figure 52). For simplicity suppose $y = 2$ when $t = 0$. For $\Delta t = 0.1$ s, $\Delta y \approx 5 \times 2 \times 0.1 = 1$.

So y becomes $2+1 = 3$ approximately at a time $t = 0.1$ s.

For the next interval of 0.1 s, $y = 3$ and hence $\Delta y \approx 5 \times 3 \times 0.1 = 1.5$.
At a time of 0.2 s, y reaches the value 4.5, approximately.
For the next interval 0.1 s, $y \approx 4.5$ and hence $\Delta y \approx 5 \times 4.5 \times 0.1 = 2.25$.
At a time of 0.3 s, y reaches a value of 6.75, approximately.

And so the graph builds up as the diagrams in figure 52 show. The graph should be plotted as the calculations proceed so that any errors of calculation may be shown up.

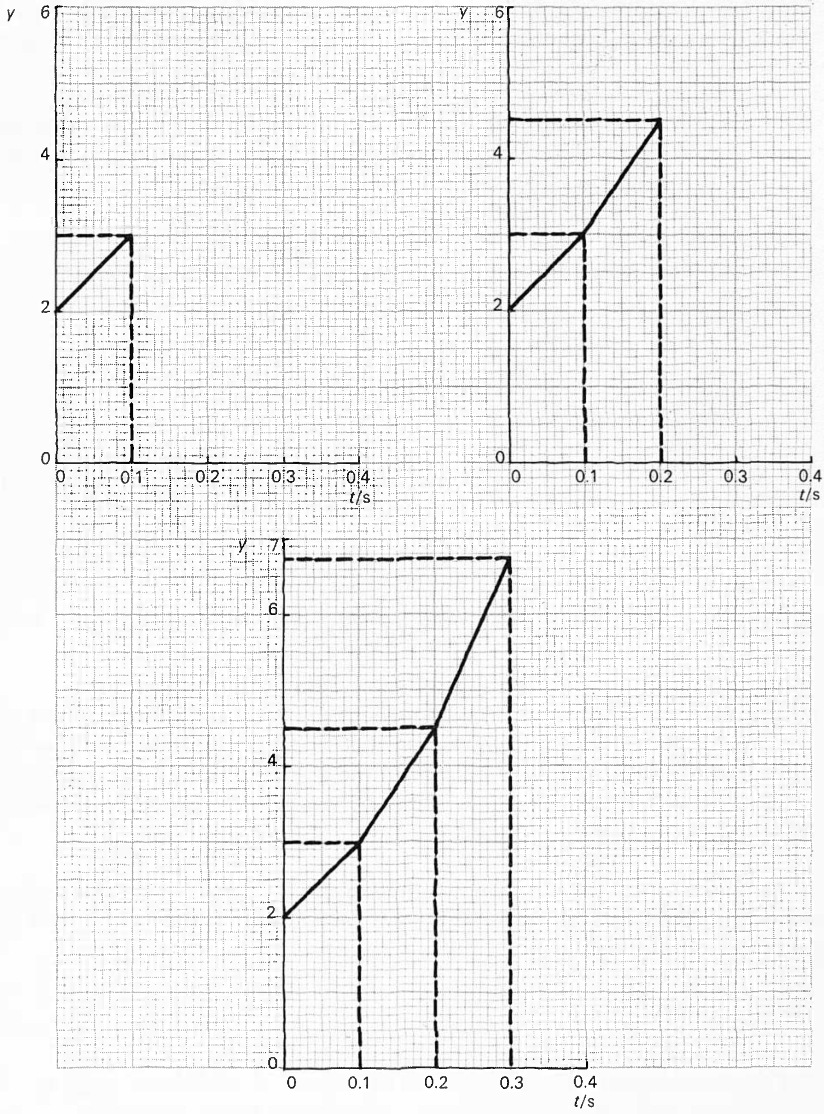


Figure 52

It would probably be best not to worry the class with the mathematical reasons why Δy is only approximately equal to $5y\ \Delta t$. The limiting value of $\Delta y/\Delta t$ as Δt tends to zero is dy/dt ($= 5y$ in this case). For finite values of Δt, dy/dt is only approximately equal to $\Delta y/\Delta t$, the approximation becoming better the smaller Δt becomes. Students could be shown the graphical interpretation of this (figure 53):

$m\ \Delta t = (dy/dt)\Delta t$ is not exactly equal to Δy, but it is probably enough simply to say that in the real graph, y changes continuously and not in a series of short straight lines.

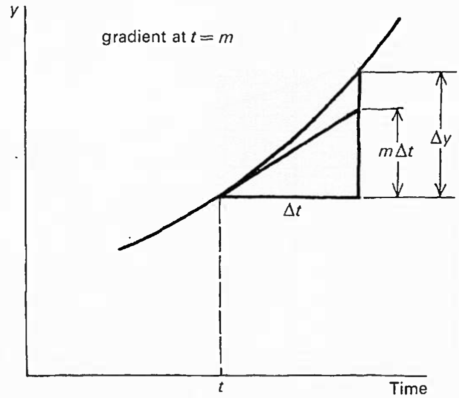


Figure 53

Solution of $d^2y/dt^2 = 10$

How does y vary with t in the case of a variation described by $d^2y/dt^2 = K$, a constant?

d^2y/dt^2 means that differentiation with respect to t has been carried out on y (so getting the gradient of the y–t graph) and then again on the expression for the gradient. It stands for the rate of change of the gradient of the y–t graph with t. If this step is not apparent to students it may be worth while talking in terms of velocity and acceleration, as follows.

For constant acceleration, the velocity–time graph is a straight line and the gradient equals the acceleration, i.e. $dv/dt = K$. In incremental form $\Delta v/\Delta t = K$. But v is the slope of the s–t graph, so that

$$\frac{\Delta(\text{slope of } s\text{–}t \text{ graph})}{\Delta t} = K.$$

$d^2y/dt^2 = K$ means that the rate of change of the slope of the y–t graph is constant. Two more pieces of information are needed before the graph can be drawn: *1* where to start drawing it, *2* what its initial gradient is, as was found when 'reverse differentiation' was used (section 7.1).

Consider the case of an object falling from rest in the Earth's gravitational field. The acceleration is $10\ \text{m s}^{-2}$, i.e. for a distance–time graph, $\Delta(\text{slope}) \approx 10\ \Delta t$ and for a time interval $\Delta t = 0.1$ s, $\Delta(\text{slope}) \approx 1\ \text{ms}^{-1}$.

The slope changes in each successive time interval of 0.1 s by 1 m s^{-1} and the slope at $t = 0$ is zero. Thus (figure 54) the slope is 1 m s^{-1} at $t = 0.1$ s and a line of this slope must be drawn extending from $t = 0.05$ s to $t = 0.15$ s and rising from $y = 0$ to $y = 0.1$ m. This is the sensible thing to do; a line drawn with this slope from the origin would mean that the object had a velocity of 1 m s^{-1} at $t = 0$, and if drawn from $t = 0.1$ s, it would mean that there was no movement for the first tenth of a second. In the second interval of 0.1 s (from $t = 0.15$ s to $t = 0.25$ s) the slope is 2 m s^{-1}, and in this time y increases by 0.2 m to become 0.3 m. In the third interval of 0.1 s (from $t = 0.25$ s to $t = 0.35$ s) the slope is 3 m s^{-1} and y increases to 0.6 m, and so on. Figure 54 shows how the graph builds up.

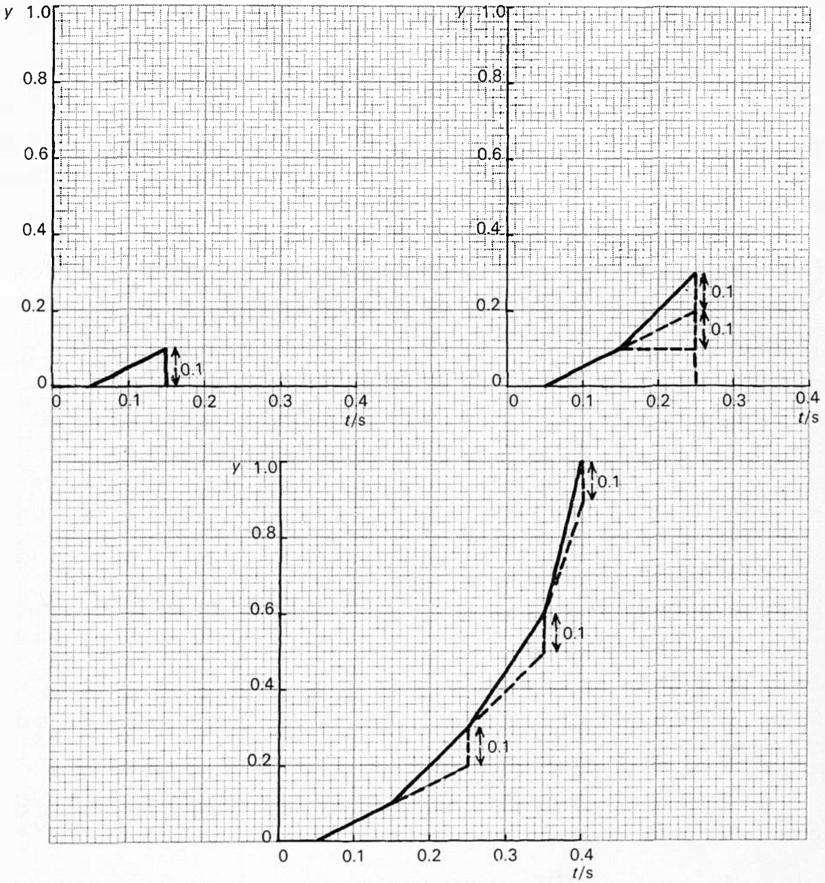


Figure 54

Examples for section 7.2

Students should be urged to tabulate their calculations in this work.

1 Plot a graph of distance s against time t for a car whose velocity v is given by the equation

$$v = \frac{ds}{dt} = 2 - 5t \text{ m s}^{-1}.$$

The initial value of s is 2 m.
When is s a maximum and when is s zero?

[0.40 s and 1.38 s.]

2 Obtain a graph of y against time t if $dy/dt = -2y$ and if the value of y for $t = 0$ is 2, over a range of time of 1 s. Estimate the value of t which makes y equal to 0.6.

[0.55 s.]

3 In a number of cases, the force F acting on an object is found to vary inversely as the square of the distance r of the object from a certain point, i.e. $F = k/r^2$. For a small change of r, Δr, the energy converted to potential energy, ΔE, is given by $\Delta E \approx -F\,\Delta r$.

Convert this expression into a differential equation involving dE/dr, k, and r only, and solve it for the case where $k = 10^{-7}$ N m^2 and where $E = 10^{-6}$ J at $r = 0.1$ m. Use a range of r values from 0.10 m down to 0.01 m with $\Delta r = 0.01$ m. Show results graphically.

Range of r/m	0.105–0.095	0.095–0.085	0.085–0.075	...
Δr/m	-0.01	-0.01	-0.01	...
Force at mean value of r/N	10^{-5}	1.23×10^{-5}	1.56×10^{-5}	...
$\Delta E = -F \times \Delta r$ J	$+10^{-7}$	$+1.23 \times 10^{-7}$	$+1.56 \times 10^{-7}$	...
Energy change/J *from*	9.50×10^{-7}	10.50×10^{-7}	11.73×10^{-7}	
to	10.50×10^{-7}	11.73×10^{-7}	13.29×10^{-7}	...

Table 9

4 Plot the distance–time graph for a particle whose acceleration is described by the equation $d^2s/dt^2 = 3$.
Initially $s = 0$ at $t = 0$ and the particle is at rest.

5 Plot the distance–time graph for a particle oscillating on the end of a vertical spring. The motion is described by

$$\text{acceleration} = \frac{d^2x}{dt^2} = -0.1\,x.$$

Initially, at $t = 0$, $x = 100$ units and the velocity is zero.
Take $\Delta t = 1$ s and plot the graph for 20 s.

6 Try problem **5** with a positive sign instead of a negative sign in the equation, i.e.

$$\frac{d^2x}{dt^2} = +0.1\,x.$$

7 A rocket, whose mass is m at some instant, uses up a mass Δm of fuel in time Δt and so increases its speed by Δv. The fuel gases are ejected at velocity u with respect to the rocket.
The average force on the rocket over time Δt = mass × acceleration = $m\,\Delta v/\Delta t$.
This equals the rate of change of momentum of the fuel ejected.

Thus $$m\frac{\Delta v}{\Delta t} = -u\frac{\Delta m}{\Delta t}.$$

If the initial mass of the rocket is M_0 and fuel is burnt at a constant rate, μ, then after time t, $m = M_0 - \mu t$. Thus

$$(M_0-\mu t)\frac{\Delta v}{\Delta t} = u\mu \quad \text{or} \quad \frac{\Delta v}{\Delta t} = \frac{u\mu}{M_0-\mu t}.$$

a Assuming M_0 = 10 000 kg, μ = 50 kg s^{-1}, and u = 2000 m s^{-1}, draw a graph showing how the acceleration of this rocket varies with time in the first 50 s.
The above would be true for the rocket moving in space where there was no gravitational field. If the rocket is lifting off from Earth, gravity reduces all the accelerations shown on the graph by 9.8 m s^{-2} (assuming the rocket doesn't get far enough away for the acceleration due to gravity, g, to change).

Net acceleration $$= \frac{u\mu}{M_0-\mu t} - g.$$

b Now plot a distance–time graph for the first 50 s of flight. Use Δt = 10 seconds and the average acceleration obtained from the figures of **a** after subtracting 9.8 m s^{-2}.

The graph is an approximate solution of $$\frac{d^2s}{dt^2} = \frac{u\mu}{M_0-\mu t} - g.$$

7.3 Areas and integration

Students will need to be reminded that there is sometimes significance in the area 'under' a graph, the area being bounded by the horizontal axis, the graph, and two verticals. Revision of its measurement is probably best achieved by means of a simple exercise.

Revision exercise for section 7.3

The graphs in figure 55 represent patterns of values obtained as measurements in certain circumstances. By considering gradients and areas, what can you deduce about those circumstances?

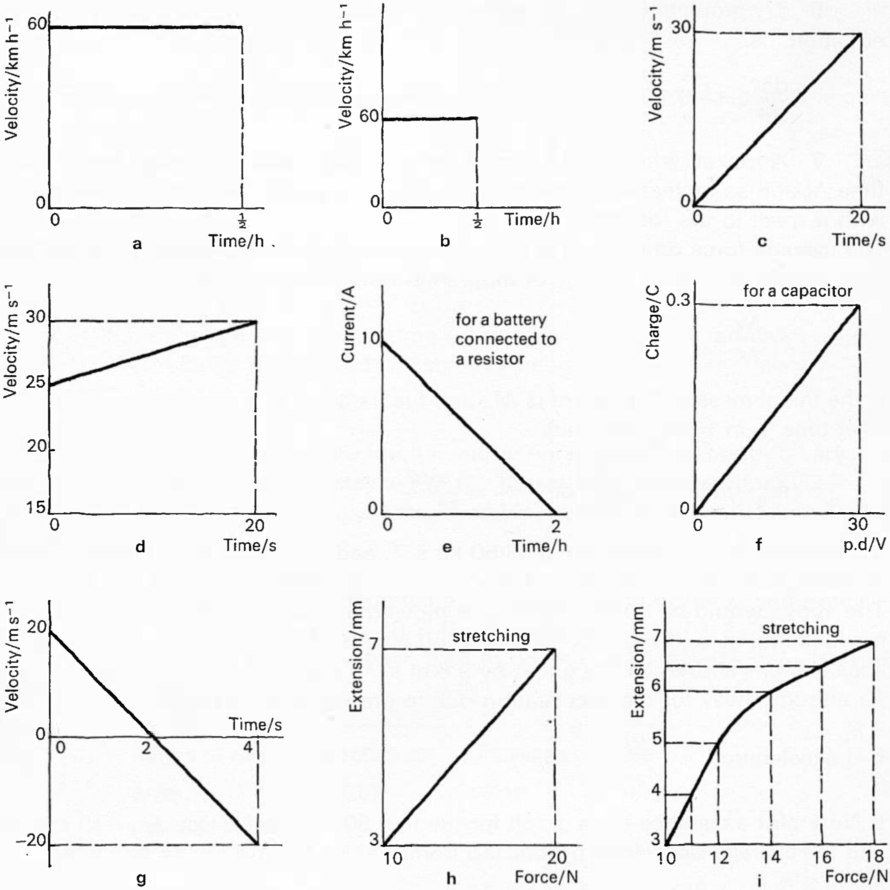


Figure 55

[Figure 55 *a* Constant velocity, distance travelled = 30 km.
b Constant velocity, distance travelled = 30 km (same 'area').
c Constant acceleration 1.5 m s^{-2}, distance travelled = 300 m.
d Constant acceleration 0.25 m s^{-2}, distance travelled = 550 m (false 'origin').
e Discharging battery. Charge capacity = 36 kC.
f Energy stored = area to left of graph = 4.5 J.
g Motion under gravity, acceleration = -10 m s^{-2}. Reaches maximum height (in 2 s) of 20 m. Returns to starting point in 4 s (distance travelled = 0). Area below time axis must be counted negative.
h Work done in extending from 3 mm to 7 mm is 0.06 J. Graph has false origin. Area is measured to left of line.
i Work done in extending from 3 mm to 7 mm is 0.051 J. Graph has false origin. Area is measured to left of curve.]

It is by convention that the phrase 'area under a graph' is used. Of course, if the variables were plotted 'the other way round', it would be the appropriate area to the left of the graph which was significant. The graphs shown in figure 55 include cases to illustrate this (figure 55 *f, h,* and *i*).

Students must realize that the areas are to be measured by obtaining dimensions from the scales [figure 55 *a* and *b*], beware of 'false' origins [*d, h, i*], and be able to find the appropriate area by the method of strip division [*i*]. The case where areas are given negative signs might be discussed [*g*].

Those who need more practice at finding areas by strip division might be set the task of finding the area of a quadrant of a circle of radius 50 mm by this method [1975 mm² for strips 5 mm wide] and use it to find a value for π [3.16].

This method of obtaining an area gives an approximate answer. How could that answer be made more accurate? (Use strips of smaller width.) Mathematicians would say that, for a graph of y against x, a strip of width Δx had been chosen and

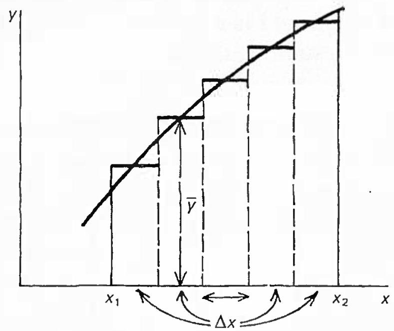


Figure 56

then the area of that strip was $\bar{y}\Delta x$, $\bar{y}$ being the mean value of y for that strip. To show that the areas of several strips have been added together, the symbol $\sum$ (sigma) is used and, if the two verticals are drawn at x_1 and x_2, this is indicated by writing them below and above the $\sum$.

$$\sum_{x_1}^{x_2} \bar{y}\Delta x$$

stands for the area calculated by the sum of the areas of strips Δx wide between the axis, the curve, and the verticals at x_1 and x_2.

To find the exact area, the size of the strip width must be made very small, i.e. Δx must tend to zero. Then the sum will get closer and closer to the true area as Δx becomes smaller. Mathematicians would then write

$$\text{area} = \int_{x_1}^{x_2} y \, dx,$$

'the integral of y with respect to x'. The sign $\int$ is simply an old English 's' standing for the words 'sum of' and the 'dx' indicates that the strip width tends to zero.

Again, x_1 and x_2 indicate the limits between which the area is measured. For example, if we wished to express the fact that the velocity gained by a body, in the time between 0 and t, was the area under an acceleration–time graph, we should write this as:

$$\text{Velocity gained} = \int_0^t a\,dt.$$

This is equal to $(v-u)$ where v is the velocity at t seconds and u that at 0 seconds.

$$\text{Thus} \quad v = u + \int_0^t a\,dt.$$

For a = constant, the result has been obtained before (section 7.1). 'Reverse differentiation' gave $v = u+at$ and so, as the results must be the same:

$$\int_0^t a\,dt = at.$$

It is not difficult to interpret this piece of mathematical 'shorthand'. Obviously, the area under the curve of a against t is $a\Delta t + a\Delta t + a\Delta t \ldots$. This equals $a(\Delta t + \Delta t + \Delta t \ldots)$ which, in turn, equals at.

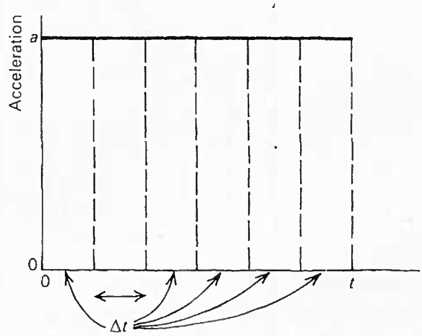


Figure 57

Teachers should take the argument a step further by discussing distance travelled as the area under a velocity–time graph for motion under constant acceleration. That gives:

$$\text{Distance travelled} = \int_0^t v\,dt = ut + \tfrac{1}{2}at^2 \text{ by the area method}$$

or $$\int_0^t v\,dt = \int_0^t (u+at)\,dt = ut + \tfrac{1}{2}at^2 \text{ by reverse differentiation.}$$

Again, the result wanted is obtained by the reverse differentiation of the expression being integrated.

Is it always so? Do 'integration' and 'reverse differentiation' mean the same thing? Teachers could stop here and say the processes are the same, but for those students who want a little more, the following general treatment might be satisfactory.

Reverse differentiation is the process by which, given the graph of the gradient of a y–x curve for different values of x, the *shape* of the y–x curve could be found. There is a whole family of possible curves (figure 58), each differing from the others only by a constant. The constant is the value of y when $x = 0$, for example. If the constant can be found, reverse differentiation will reveal how y changes with x; it will give the equation relating y to x.

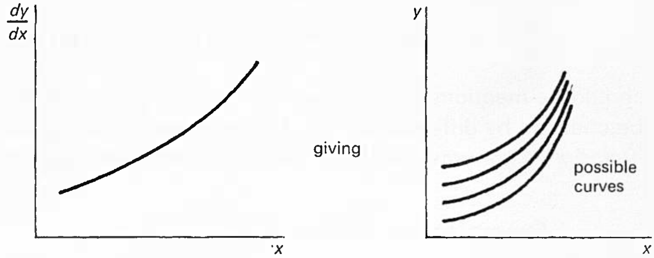


Figure 58

Finding the area under a $(dy/dx) - x$ graph between 0 and some value x means adding up strip areas when the width tends to zero (figure 59). The height of each strip is $\Delta y/\Delta x$ and the width is Δx, so each strip has an area Δy. If the strip widths tend to zero, and the area from 0 to x is calculated, then we have found the total change of y between 0 and x.

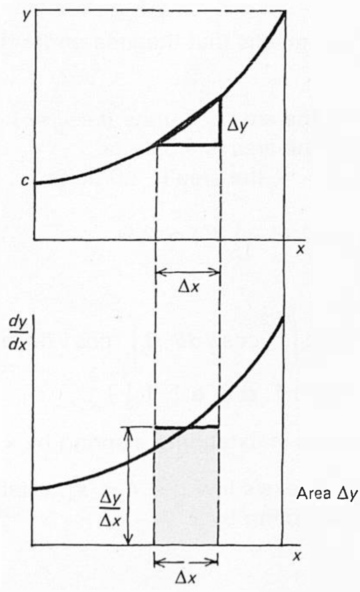


Figure 59

$$\int_0^x \frac{dy}{dx}\,dx = y - c$$

or $$y = \int_0^x \frac{dy}{dx}\,dx + c.$$

Again this is the equation relating y to x once c is known. The two processes are the same, the term integration being used in preference to 'reverse differentiation'. The class should be told that, in practice, the limits are not stated, it being understood that $\int \ldots dx$ stands for $\int_0^x \ldots dx$.

The general rule that

if $$\frac{dy}{dx} = x^n, \quad \text{then } y = \frac{x^{n+1}}{n+1} + \textit{constant} \quad \text{(except when } n = -1)$$

should be mentioned, as should the fact that the result of an integration can always be checked by differentiating it. The method of dealing with limits which are not '0 and x', e.g. numerical limits, should be apparent from the examples that follow.

Examples for section 7.3

1 Evaluate the following and check by differentiating.

a $\int_0^t 2\,dt$ **b** $\int_0^t -2t^{-2}\,dt$ **c** $\int_0^t 6t^2\,dt$ **d** $\int_0^x x^4\,dx$ **e** $\int_0^y (2+4y-y^3)\,dy$ **f** $\int_0^F dF$ **g** $\int_0^x -x\,dx$.

[**a** $2t$; **b** $2/t$; **c** $2t^3$; **d** $\frac{1}{5}x^5$; **e** $2y+2y^2-\frac{1}{4}y^4$; **f** F; **g** $-\frac{1}{2}x^2$.]

2 $\int_0^x x\,dx = \frac{1}{2}x^2$ means that the area under the graph of $y = x$ from $x = 0$ to x is $\frac{1}{2}x^2$.

From $x = 0$ to $x = 4$, the area is 8 units (i.e. $\frac{1}{2}4^2$).

From $x = 0$ to $x = 3$, the area is 4.5 units.

Thus from $x = 3$ to $x = 4$, the area is 3.5 units,

i.e. $$\int_{x=3}^{x=4} x\,dx = (\tfrac{1}{2}4^2 - \tfrac{1}{2}3^2) = 3.5.$$

Find the values of:

a $\int_{t=2}^{t=4} 2t\,dt$ **b** $\int_{-2}^{3} x^2\,dx$ **c** $\int_0^{\pi/2} \cos\theta\,d\theta$ **d** $\int_0^{\pi} \cos\theta\,d\theta$ **e** $\int_0^{\pi/2} \sin\theta\,d\theta$ **f** $\int_1^2 \frac{1}{x^2}\,dx$

[**a** 12; **b** $11\frac{2}{3}$; **c** 1; **d** 0; **e** 1; **f** $\frac{1}{2}$]

3 The work done in extending a spring by x is $\int_0^x F\,dx$.

If the spring follows Hooke's law [i.e. $F \propto x$], establish an equation for the work done in extending the spring by x.

[$\frac{1}{2}Fx$.]

4 Water leaks through a dam wall at a rate given by

$$\frac{dV}{dt} = 10+3t^2 \text{ dm}^3 \text{ s}^{-1}.$$

How much water will have leaked away in 10 seconds?

[1100 dm^3.]

5 The rate at which the electric potential V varies with the distance r from the centre of a sphere carrying charge Q is represented by the equation

$$\frac{dV}{dr} = -\frac{Q}{4\pi\varepsilon_0 r^2}.$$

Integrate this to find how the potential V varies with distance.

$$\left[V = \frac{Q}{4\pi\varepsilon_0 r} + \text{a constant.}\right]$$

Section 8

Exponential variations

8.1 Introduction

Exponential variations are first discussed in connection with the decay of charge on a charged capacitor when a resistor is connected across it (in Unit 2, *Electricity, electrons, and energy levels*), and then in connection with the decay of activity of a radioactive element (in Unit 5, *Atomic structure*). The *Students' books* for these Units contain examples of a variety of exponential changes to show that it is a pattern worth investigating. Groups of students could tackle more of these problems, different from those they attempted during the physics lessons. Teachers should use as many or as few of these as they think necessary to give students a 'feel' for this kind of variation.

After the problems have been tried, a discussion to bring out the major points regarding exponential changes would be useful. All the curves have properties similar to $N \propto 2^t$ where t is time measured in units of the doubling time (or to $N \propto 2^{-t}$ where t is now in units of halving time). N increases (or decreases) by a constant factor in equal time intervals and the curve could be termed a constant ratio curve. The word 'exponential' should be introduced to describe this type of variation (figure 60).

The gradient of an exponential curve, dN/dt, is directly proportional to N. This comes out of the fact that dN/dt is also an exponential curve with respect to t (see examples **1**, **2**, and **3**). If $N \propto 2^t$, then $dN/dt \propto 2^t$ and so $dN/dt \propto N$.

Summarizing, if N varies exponentially with t, then

1 N changes by a constant factor in equal time intervals.

2 The gradient dN/dt also changes by the same constant factor in those equal time intervals (i.e. dN/dt is exponential too).

3 The gradient dN/dt is proportional to N at every moment, i.e. if N is doubled (or halved), dN/dt is doubled (or halved).

Examples for section 8.1

1 Cells which reproduce by dividing into two may, as long as the food supply is adequate, double their number at regular intervals.

Time t	0	1	2	3	4	5	units
Number N	1	2	4	8	16	32	units

a What is the number of cells after t units of time?

[2^t.]

b Draw a graph of N against t.

c What equation represents this pattern of variation?

[$N = 2^t$.]

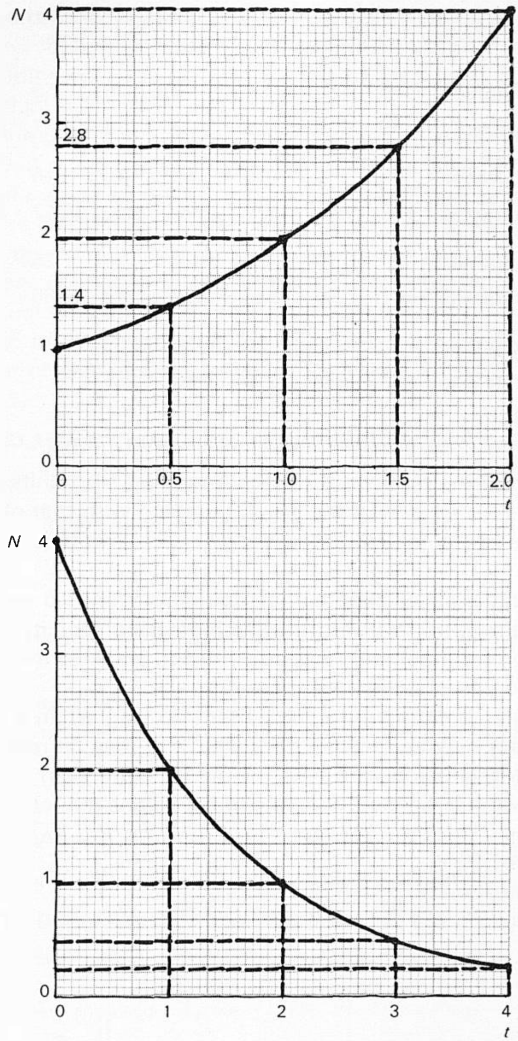


Figure 60

d Find the ratio of N at each unit of time to its value at the time 1 unit before, e.g.

$$\frac{N \text{ at time 2 units}}{N \text{ at time 1 unit}} = \frac{4}{2} = 2.$$

[2.]

What do you notice about these ratios?

[All equal.]

e Obtain values of $\Delta N/\Delta t$. How does $\Delta N/\Delta t$ vary with time t? How does it vary with N?

[$\Delta N/\Delta t$ varies with t in the same manner as N does. That is, $dN/dt \propto N$.]

2 The data in table 10 give the weekly death toll in the Great Plague, beginning with the week ending 25 April 1665 (week 1).

Week number	1	2	3	4	5	6	7	8	9	10
Plague deaths	0	0	9	3	14	17	43	112	168	267
Week number	11	12	13	14	15	16	17	18	19	20
Plague deaths	470	725	1089	1843	2010	2817	3880	4237	6102	6988
Week number	21	22	23	24	25	26	27	28	29	30
Plague deaths	6544	7165	5533	4929	4327	2665	1421	1031	1414	1050
Week number	31	32	33	34	35					
Plague deaths	652	333	210	243	281					

Table 10

From Creighton, C. (1965) A history of epidemics in Britain, *Volume I, Cass.*

a Draw a graph of the number of deaths N against t (in units of weeks).
b How does the earlier part of the graph compare with that of $N = 2^t$ in example **1**?
c Obtain values of $\Delta N/\Delta t$. How does $\Delta N/\Delta t$ vary with time t? How does it vary with N?

[The initial rise is exponential, doubling every 1.5 weeks. This rise is not maintained, partly because of an increasing scarcity of victims.]

d Find the ratio of the number of deaths in each week to the number in the week before. What do you notice about these ratios?

3 An experiment in which the current flowing when a capacitor discharges through a resistor may be recalled from Unit 2 or may be repeated. See experiment 2.17. The current variation with time (i.e. at 10 s intervals) is recorded from the moment the switch is opened. Suitable initial values are

1 a p.d. of 10 V,
2 a current of 100 μA (R = 100 kΩ),
3 a charge on the capacitor of 5×10^{-3} C (C = 500 μF).

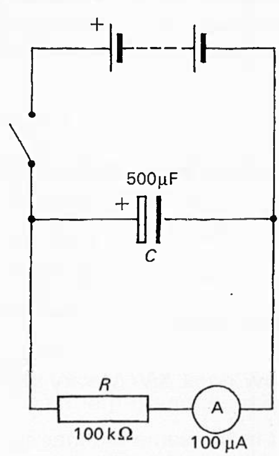


Figure 61

If the current at any moment is I, then the p.d. across $C = RI$, i.e. $V = 10^5\ I$, and hence the charge $Q = 500 \times 10^{-6} \times 10^5\ I$, so that $Q = 50\ I$.
Thus the current readings can be used to give the charge decay curve.

a Why does the current fall?

[Because the p.d. driving it falls and that falls because the previous current flow removed some charge. The current falls because of the current itself. This is similar to an epidemic but falling instead of growing. The more 'flu victims there are, the more infectious people there are about so the more new cases there are – until the virus runs out of people.]

b By what factor does Q change in equal time intervals? By what factor does dQ/dt change in equal time intervals?

[About 0.55 in 30 s. Note that $Q = 50\ I = -50\ dQ/dt$.]

4 The number of cars in private ownership in Britain has risen over the years as shown in table 11.

Year	1947	1948	1949	1950	1951	1952	1953	1954	1955	1956
Cars (in millions)	1.94	1.96	2.13	2.26	2.38	2.51	2.76	3.10	3.52	3.89
Year	1957	1958	1959	1960	1961	1962	1963	1964	1965	1966
Cars (in millions)	4.19	4.55	4.97	5.53	5.98	6.56	7.37	8.25	8.92	9.51

Table 11

Data from Central Statistical Office Annual abstracts of statistics *(1958–70). By permission of the Controller, H.M.S.O.*

a Plot a graph of the number of cars against time.

b Find the ratio of the number of cars in each year to the number in the previous year, e.g.

$$\frac{\text{Number in 1949}}{\text{Number in 1948}} = \frac{2.13}{1.96} = 1.09.$$

5 Table 12 gives the used-car prices for a Mini Saloon over a period of years.

Year of model	1968	1967	1966	1965	1964	1963	1962	1961
Used-car price/£	509	425	360	315	270	235	205	180

Table 12

From Motorists' guide to new and used car prices *(1970), Blackfriars Press.*

a Plot a graph of the price against year.

b By what factor does the price change in succeeding years? What do you notice about these ratios?

8.2 Identifying exponential variation

The teaching should now aim to produce a more general equation for exponential change.

Suppose that a quantity N varies exponentially with time, that at $t = 0$ the value of the quantity is N_0, and that in 1 second, it changes by a factor a. A table can be drawn up (table 13) showing how the quantity changes with time:

Time in seconds	0	1	2	3	4	5
Value of the quantity	N_0	N_0a	N_0a^2	N_0a^3	N_0a^4	N_0a^5

Table 13

Clearly, after time t, the quantity has a value given by

$$N = N_0\,a^t.$$

Note that a is simply a number.

Students could be shown that this has the general property of an exponential change, and that in equal time intervals (no matter what that interval is), N changes by a constant factor. If $N_{t+\Delta t}$ is the value of N at $(t+\Delta t)$ and N_t is its value at t, then:

$$\frac{N_{t+\Delta t}}{N_t} = \frac{N_0\,a^{(t+\Delta t)}}{N_0\,a^t} = \frac{a^t a^{\Delta t}}{a^t} = a^{\Delta t}$$

and this depends only on the time interval Δt and not on t. For equal time intervals, the ratio of the N value at the end of the interval to that at the beginning is constant.

Exponential patterns can also be represented by an equation written in terms of the 'doubling time'. Then it would be

$$N = N_0\,2^x$$

where x is the time measured in units of 'doubling time'. This is really the same equation as $N = N_0\,a^t$ (where a is the factor of increase occuring in 1 second). Suppose the 'doubling time' was T, then $xT = t$ and,

$$N = N_0\,a^t = N_0\,a^{Tx} = N_0\,(a^T)^x = N_0\,2^x$$

giving $a^T = 2$, as it should do, for the ratio of the values of N at the end and beginning of a time interval of Δt is $a^{\Delta t}$. Thus, for exponential change, the expression describing that change is

$$\underbrace{N}_{\substack{\text{value at the}\\ \text{time considered}}} = \underbrace{N_0}_{\substack{\text{value at}\\ \text{the start}\\ t=0}} \times \underbrace{a}_{\substack{\text{factor of change}\\ \text{in 1 unit of time}}}{}^{t \rightarrow \text{ number of time units elapsed}}$$

It is not always easy to use the method of finding the ratio of N at the end to N at the beginning of several equal time intervals to decide whether data fit this type of equation. Fluctuations in the data or inaccuracies of measurement cause fluctuations in the value of the factor of increase. It would be easier to recognize an exponential variation if the data could be plotted in such a way as to give a straight-line graph.

Those who have remembered that if

$$\frac{N}{N_0} = a^t, \text{ then } t = \log_a \frac{N}{N_0}$$

will not find it difficult to take logarithms to obtain

$$\lg N = \lg N_0 + t \lg a.$$

Others might be led to this by being reminded how a straight line graph could be obtained from data fitting $p = Cq^m$ (section 4.3). Thus, by plotting a lg N–t graph, a trend away from exponential variation could be recognized, and if exponential, a and N_0 could be determined from the gradient and intercept respectively. Students will need practice at handling data in this way and might now re-examine the data of the introductory questions or attempt questions similar to those which follow. Question 4 is a good introductory question for the next section, and if the graphs are plotted now, they should be saved.

In discussing equations for exponential change, teachers should not omit decreasing changes. For example, if the quantity decreases by $1/a$ in 1 second, then, after t seconds,

$$N = N_0\left(\frac{1}{a}\right)^t = N_0\frac{1}{a^t} = N_0 a^{-t}.$$

The 'halving time' or 'half-life', T, is such that $a^T = 2$.

Examples for section 8.2

1 Is the following an exponential change?

The data in table 14 relate to the power capacity of electrical generating plant in Great Britain measured in GW (10^9 W).

Year	1947	1948	1949	1950	1951	1952	1953	1954	1955	1956
Capacity/GW	12.9	13.1	13.8	15.0	16.2	17.7	19.2	20.6	22.5	24.6

Year	1957	1958	1959	1960	1961	1962	1963	1964	1965	1966
Capacity/GW	26.6	28.0	30.0	31.9	33.9	37.2	39.3	40.0	43.9	46. 2

Table 14

Data from Central Statistical Office (1970) Annual abstract of statistics. *By permission of the Controller, H.M.S.O.*

[This rises exponentially except for some minor pauses.]

2 The world population has been estimated as in table 15. Does this show an exponential change?

Year	1650	1750	1800	1850	1900
Population	0.47×10^9	0.69×10^9	0.92×10^9	1.09×10^9	1.57×10^9
Year	1920	1930	1940	1950	1960
Population	1.81×10^9	2.01×10^9	2.25×10^9	2.51×10^9	2.99×10^9

Table 15

From Thompson, W. S. (1965) 5th revised edition, Population problems, *McGraw-Hill.*

[This rise is more than exponential.]

3 In an experiment in which a capacitor discharges through a resistor, the initial current I was 100 μA. Its value was recorded at 10 s intervals and found to change as follows:

60 μA, 36 μA, 22 μA, 13 μA, 8 μA, 5 μA.

Is this an exponential change? If so, by what factor does the current change in 1 s (i.e. what is a in $I = I_0 a^t$)? What is the halving time?

[Yes; $a = 0.95$; 13.7 s.]

4 a Use logarithm tables to plot graphs of $N = 2^t$ and $N = 3^t$ for values of t from 0 to 1.0 units.

b These curves are exponential: for equal increase of t, N increases by a constant factor. From your answers to **a** find the values of $2^{0 \cdot 25}$.
Knowing that $2^{3 \cdot 0} = 8.0$, what is $2^{3 \cdot 25}$?

[1.19; 9.52.]

c Explain why $N = a^t$ will have the property stated in **b** no matter what value the number a has.

Note

There are many cases (some are included in the above examples) where the quantities concerned are not continuous variables. In plotting the data of the above examples, some students may produce a straight line chart where N can only take on integral values (figure 62). This is probably a better way of displaying the data than plotting the points and drawing a curve through them.

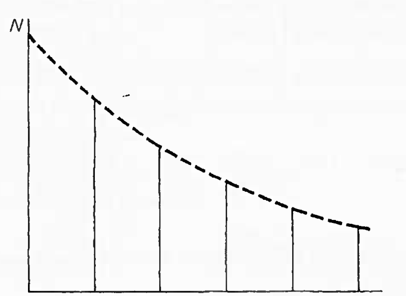


Figure 62

When an equation such as $N = N_0 a^t$ is used to represent the variation of a discontinuous variable, N has to be restricted to possible values.

8.3 The differential equation

This section deals with the integration of $dN/dt = kN$. The work is part of the Physics course and is thoroughly treated there, but for those who wish to follow the argument again, a version of the work covered in the chapter 'Exponential changes' in the *Students' book* for Unit 5 follows.

The gradient of an exponential curve also varies exponentially. If students need a reminder of this, it can be provided quickly by calculating the gradient of $N = 10^t$ [$t = \lg N$] using a table of anti-logarithms as in table 16.

t	N	$t+\Delta t$	$N+\Delta N$	ΔN	$\Delta N/\Delta t$	
0.00	1.000	0.01	1.023	0.023	2.3	
						$\frac{3.7}{2.3} = 1.60$
0.20	1.585	0.21	1.622	0.037	3.7	
						$\frac{5.8}{3.7} = 1.57$
0.40	2.512	0.41	2.570	0.058	5.8	
						$\frac{9.3}{5.8} = 1.60$
0.60	3.981	0.61	4.074	0.093	9.3	
						$\frac{14.7}{9.3} = 1.58$
0.80	6.310	0.81	6.457	0.147	14.7	
						$\frac{23.3}{14.7} = 1.59$
1.00	10.000	1.01	10.233	0.233	23.3	

Table 16

Clearly dN/dt is also exponential and hence $dN/dt \propto N$ is true for exponential changes. It is the latter form of variation which is first met in Unit 2 in the case of the discharge of a capacitor ($dQ/dt \propto Q$).

The question to be solved is: if $dN/dt = kN$, how does N vary with t? The simplest case (i.e. $dN/dt = N$) should be taken first and the solution obtained by a numerical method.

Students might be asked: 'Using the same axes as you did for question 4, section 8.2, plot out, step by step, the growth of N starting from $N = +1.0$ at $t = 0$ if $dN/dt = N$. Take 0.1 s steps for Δt and stop when you reach $t = 1.0$ s.' Since $\Delta N \approx N\,\Delta t$, for the first time interval of 0.1 s, $\Delta N \approx 0.1$ and N becomes 1.1.

For the next 0.1 s, $\Delta N \approx 1.1 \times 0.1 = 0.11$ and N becomes 1.21, and so on. This is a little larger because N has increased. At $t = 1.0$ s, N will have risen to about 2.6 (figure 63).

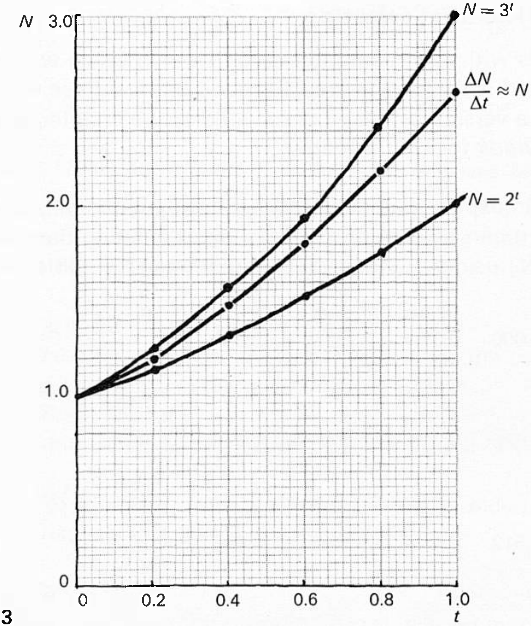


Figure 63

If smaller time intervals than 0.1 s are used, N rises to a higher value (why?) nearer to 2.7. Accurate work gives 2.718 28 . . . as the value N reaches when $t = 1.0$ s, if it has the value 1.0 initially. If $N = a^t$ is the equation for an exponential curve (so that $N = 1$ when $t = 0$), then N reaches the value a when $t = 1$. In the case where $\mathrm{d}N/\mathrm{d}t = N$, a has to have this value 2.718 . . ., for which the symbol e is used [i.e. $N = \mathrm{e}^t$].

'Now repeat the curve drawing for the growth represented by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0.7\,N.\text{'}$$

The graph obtained is very close to that for $N = 2^t$.

'Use logs to find what $\mathrm{e}^{0 \cdot 7}$ is or estimate its value from the first curve.'

The calculated value is 2.01; from the graph, 1.95. Instead of $N = 2^t$ being written for the solution of $\mathrm{d}N/\mathrm{d}t = 0.7\,N$, this could be written as:

$$N = (\mathrm{e}^{0 \cdot 7})^t = \mathrm{e}^{0 \cdot 7t} \qquad \text{if } N = 1.0 \text{ when } t = 0.$$

'What do you think N would be equal to if $\mathrm{d}N/\mathrm{d}t = kN$ and N was equal to 1.0 when $t = 0$?'

In the previous example, k was 0.7 and N came out to be $e^{0.7t}$. Generally, it might be expected that $N = e^{kt}$ would give $dN/dt = kN$. It is worth emphasizing that this is not different from a^t; it is simply that a has to have the value e^k if the gradient of the curve (dN/dt) at a particular time is to be k times the value of N at that time.

It is not difficult to extend this to the case where the variation starts at some value other than 1.0, say N_0. A numerical approach might be easier but perhaps the following argument might be appreciated. Compare $M = e^{kt}$ for which $M = 1.0$ when $t = 0$ with $N = N_0 e^{kt}$ for which $N = N_0$ when $t = 0$.

Clearly, at any chosen time, $N = N_0 \times M$ and in a small time interval, $\Delta N = N_0 \times \Delta M$.

Thus the gradient, dN/dt, of the N–t graph will be N_0 times larger than the gradient, dM/dt, of the M–t graph,

that is, $\dfrac{dN}{dt} = N_0 \times \dfrac{dM}{dt}$. But, $\dfrac{dM}{dt} = kM$ and $N = N_0 \times M$.

$$\therefore \frac{dN}{dt} = N_0 kM = kN.$$

The differential equation describing $N = e^{kt}$ is the same as that describing $N = N_0 e^{kt}$. Consequently, if it is known that $dN/dt = kN$ then, in general:

$$\boxed{N = N_0 e^{kt}}$$

This determines the fractional rate of growth.

This is the value of N at the start (i.e. when $t = 0$).

Examples for section 8.3

1 A bucket has a hole in the bottom, and when it is full, water leaks out at a rate of 100 $cm^3\ s^{-1}$. The rate of leakage as a function of time is shown in table 17.

Time/s	Rate/$cm^3\ s^{-1}$
0	100
10	78
20	60
30	47
40	36
50	28

Table 17

Is this an exponential variation?
[Yes.]

What is the rate of leakage when $t = 90$ s?

[10 $cm^3\ s^{-1}$.]

If the volume remaining at $t = 90$ s is 390 cm^3, how much water was there at the beginning?

[3900 cm^3.]

2 How do the graphs of the equations $dN/dt = +kN$ and $dN/dt = -kN$ differ?

3 Suppose y represents the number of families who have central heating in their homes at any time. What would the symbols Δy and $\Delta y/\Delta t$ stand for?

It might be plausible to think that a family would not consider installing central heating unless they had met other families like themselves who had done so and who recommended it. Why might a mathematical model like $\Delta y/\Delta t = ky$ be appropriate for the number of families having central heating?

4 Suppose the population of Great Britain would stay steady if there were no emigration or immigration, then suppose that immigration is stopped totally while, in each year, 10 per cent of the population of the country at the start of that year decide to emigrate within the year.

Draw a graph of the variation of population over the next 10 years, assuming that the initial population is 50 million people.

5 The p.d. V across a capacitor of capacitance C having charge Q is given by $V = Q/C$. When the capacitor is discharging through a resistance R, the current when the p.d. is V is $I = V/R$, so that $I = Q/CR$.

In a small time interval Δt, short enough for the current to be considered constant, the charge changes by ΔQ.

$$\Delta Q \approx -I\,\Delta t = -\frac{Q}{CR}\Delta t.$$

The minus sign indicates that this is a discharge, i.e. that Q falls when current flows so that ΔQ is a negative quantity.

a If $CR = 50$ s, plot the decay graph for the charge Q using $\Delta t = 5$ s and an initial value of Q of 5×10^{-3} C.

b Your graph is an approximate solution of

$$\frac{dQ}{dt} = -\frac{1}{CR}Q.$$

Why is it approximate? What is the mathematical equation describing the graph to which it approximates?

[$Q = 5 \times 10^{-3}\, e^{-0.02t}$.]

6 Table 18 gives the number of counts recorded in 1 minute (the count-rate) at various times, when a radioactive substance was placed near a counter.

Time in hours	0.5	1.0	1.5	2.0	3.0	4.0	5.0
Count-rate/minute^{-1}	9535	8190	7040	6050	4465	3300	2430
Time in hours	6.0	7.0	8.0	9.0	10.0	11.0	12.0
Count-rate/minute^{-1}	1800	1330	980	720	535	395	290

Table 18

a Test whether the data fit the mathematical model of exponential decay.

[They do.]

b Find the half-life, i.e. the time for the count-rate to fall to $\frac{1}{2}$ of the original value.

[2.3 hours.]

c If N_0 is the number of radioactive atoms at the start and N is the number of those left at time t, then $N = N_0 e^{-kt}$. Find the value of the decay constant, k.

[8.4×10^{-5} s^{-1}.]

7 Imagine that when substance A is mixed with substance B a chemical reaction starts in which energy is released so that the mixture becomes warmer and that the rate of release of energy is proportional to how warm A and B are. This situation could be described by an equation such as $dT/dt \propto T$ where T is the temperature of the mixture and t is time. If the constant of proportionality is 6×10^{-3} s^{-1}, how long will it be before the mixture explodes if the explosion temperature is 405 K and the initial temperature is 300 K?

[50 s.]

8 When γ-rays pass through lead, they are partly absorbed. The count-rate, C, for a γ-source when a thickness of x mm of lead is interposed between the source and a GM tube, is as follows, the positions of the source and detector being kept fixed:

x/mm	6	9	12	15	18	21
C/minute^{-1}	1075	925	780	660	570	480

What mathematical equation describes this number pattern? What is the half-thickness of lead in this case?

[$C = 1500\,e^{-0.054x}$; 12.8 mm.]

9 a Write down the derivative in the following cases:

1 $y = 2e^{3x}$

2 $y = y_0 e^{-kx}$

3 $N = 7e^{(2t+1)}$

[$dy/dx = 6e^{3x} = 3y$; $dy/dx = -ky_0 e^{-kx} = -ky$;
$dN/dt = 14e^{(2t+1)} = 2N$.]

b Integrate the following equations:

1 $dy/dx = e^x$

2 $dy/dx = 5y$

3 $dy/dx = 6e^{3x}$.

[$y = e^x + c$; $y = y_0 e^{5x}$; $y = 2e^{3x} + c$.]

8.4 Napierian logarithms

If a number, say N, is equal to a^t, t is called the logarithm to the base a of N. Growth and decay variations are usually expressed in terms of e. Thus, if $N = e^{kt}$, kt is the logarithm of N to the base e. This is written as '$\ln N = kt$' and called the Napierian logarithm of N. Books of tables often include these. For variations expressed by the equation $N = N_0 e^{kt}$, taking Napierian logarithms gives $\ln N = \ln N_0 + kt$. The class could gain practice by plotting a graph of $N = e^t$ using tables and comparing it with that drawn in section 8.3.

8.5 The number e

Students may be interested in knowing how the value of e can be obtained to many decimal places, and this section is included to help teachers to answer. Students should know that e can be calculated and that, like π, it 'goes on forever'. The value found in section 8.3 was approximate because, in drawing the graph, the gradient remained constant for each time interval of $\frac{1}{10}$ second. The graph of $N = e^t$ has a gradient which increases continuously, and N grows a little more rapidly than the numerical method indicated, so that the value obtained (about 2.6) was an underestimate. The smaller the time interval used, the better the value obtained, but the process is not one to be adopted because, for example, if $\Delta t = 0.01$ s, 100 operations have to be carried out to reach $t = 1$ s, and the work is tedious. However, the result that would have been obtained can be found numerically without much difficulty by setting the graphical process out as in table 19.

The numerical integration of $dN/dt = N$ used a value of $\Delta t = 0.1$ s so that $\Delta N \approx 0.1\ N = \frac{1}{10} N$.

Time interval		**N at the start**	**ΔN**	**N at the end**
from	*to*			
$t = 0$ s	$t = 0.1$ s	1	$\frac{1}{10}$	$(1+\frac{1}{10})$
$t = 0.1$ s	$t = 0.2$ s	$1+\frac{1}{10}$	$\frac{1}{10}(1+\frac{1}{10})$	$(1+\frac{1}{10})^2$
$t = 0.2$ s	$t = 0.3$ s	$(1+\frac{1}{10})^2$	$\frac{1}{10}(1+\frac{1}{10})^2$	$(1+\frac{1}{10})^3$
....	....	....	....	....
$t = 0.9$ s	$t = 1.0$ s	$(1+\frac{1}{10})^9$	$\frac{1}{10}(1+\frac{1}{10})^9$	$(1+\frac{1}{10})^{10}$

Table 19

Up to t seconds, the value of N becomes $(1+\frac{1}{10})^{10t}$ if $\Delta t = 0.1$ s.

It should not be a hard step to extend this result to the case where $\Delta t = \frac{1}{100}$ s. After 1 s, N would be equal to $(1+\frac{1}{100})^{100}$, and this works out to be just over 2.7.

To get a better result Δt must be made smaller still. In fact, e is the limiting value of $(1+1/n)^n$ when n becomes very large, i.e. tends to infinity. There is still the problem of how to work that out but it is not worth while introducing the binomial theorem to do it. Table 20 shows values of $(1+1/n)^n$ for some values of n.

n	1	2	3	10	100	1000	10000
$(1+1/n)^n$	2	2.25	2.37	2.59	2.70	2.717	2.718

Table 20

A simpler way round the problem is to look at what N would be after t seconds:

$$N = e^t = \left(1+\frac{1}{n}\right)^{nt} \text{ when } n \text{ tends to infinity.}$$

If nt is called x, then $1/n = t/x$, so that

$$N = \left(1+\frac{t}{x}\right)^x \qquad \text{provided } x \text{ tends to infinity.}$$

The class will probably accept the fact that if this were expanded, the expansion would contain terms with t, t^2, t^3 . . . etc., which we could write as

$$N = 1+At+Bt^2+Ct^3+Dt^4 \ldots$$

Even when x tends to infinity, A, B, C don't become large (because otherwise N would be large when $t = 1$ s), but the series will go on forever. To find the values of the coefficients, it must be remembered that, for $N = e^t$, $dN/dt = N$.

$$\frac{dN}{dt} = A+2Bt+3Ct^2+4Dt^3 \ldots$$

No matter what t is, dN/dt must equal N. If $t = 0$, the expressions become $N = 1$ and $dN/dt = A$, so that $A = 1$. And the coefficient of a power of t in one expression must be the same as the coefficient of that power of t in the other. Comparison shows that

$$B = \frac{A}{2} = \frac{1}{2}, \quad C = \frac{B}{3} = \frac{1}{2\times 3}, \quad D = \frac{C}{4} = \frac{1}{2\times 3\times 4}, \quad \text{etc.}$$

Thus $$N = 1+t+\frac{t^2}{2}+\frac{t^3}{2\times 3}+\frac{t^4}{2\times 3\times 4} \ldots$$

To find e, t must be equal to 1.

$$\therefore e = 1+1+\tfrac{1}{2}+\frac{1}{2\times 3}+\frac{1}{2\times 3\times 4} \ldots$$

$$= 1+1+0.5000+0.1667+0.0417+0.0083+0.0014+0.0002+ \ldots$$

$$\approx 2.7183.$$

The magnitude of the terms rapidly decreases in size and they soon become negligible, so that calculating the value of e to many decimal places is not too laborious.

Section 9

Chance

The main points required from this section are the concept of probability in terms of the number of ways an event can occur and also in terms of the frequency with which an event occurs. These ideas are considered in Unit 9, *Change and chance.*

Useful books for students are Huff, *How to take a chance* and Weaver, *Lady Luck*. The PSSC film 'Random events' (Reference no. 900 4116–5, Guild Sound & Vision Ltd, formerly Sound Services Ltd) might be useful.

9.1 Random events – frequency and probability

The idea of frequency and the concept of probability in terms of frequency can be introduced by examining a set of data in which a random variable is involved. Some methods by which the data can be obtained are:

1 Examining the results of football matches, or cricket scores.
2 Measuring the heights or weights of members of the class.
3 Obtaining the number of counts recorded by a GM tube and scaler in 30 seconds for a weak radioactive source kept at a fixed distance from the GM tube (such that about 100 counts are recorded in $\frac{1}{2}$ minute).
4 Counting the number of words per sentence in a book.
5 Counting the number of times the letter 'e' appears per line of text.

Whichever method is chosen, the data will have to be grouped and the frequency with which results lie within a group determined. The number of groups chosen should not be too small, for that will not show the data adequately, nor too large because that will necessitate the collection of a large amount of information. Generally, some 6 to 10 groups of equal range covering the whole range of the variable will be suitable. To illustrate the development, data from the results of football matches will be used.

The First Division football results for 13 September 1969 were as shown in table 21. How many teams failed to score? How many scored one goal? The number of times an event, such as scoring one goal, occurs is known as the frequency of that event, and a frequency table is made out for the data (table 22).

First Division results, 13 September 1969

Burnley	0 : 1	Arsenal
Chelsea	2 : 2	Wolverhampton Wanderers
Coventry City	2 : 2	Crystal Palace
Everton	2 : 0	West Ham United
Manchester United	1 : 0	Liverpool
Newcastle United	0 : 1	Derby County
Nottingham Forest	2 : 1	Southampton
Sheffield Wednesday	1 : 2	Leeds United
Stoke City	4 : 2	Sunderland
Tottenham Hotspur	0 : 3	Manchester City
West Bromwich Albion	2 : 2	Ipswich Town

Table 21

Event (i.e. number of goals)	**Frequency** (i.e. number of teams)
0	5
1	5
2	10
3	1
4	1

Table 22

A useful way of representing the data is by means of a histogram. In such a diagram, the frequency is represented by the area of a rectangle, the base of which represents the group range. If the group ranges are the same, then the height is proportional to the frequency. A histogram (figure 64) should be drawn from the frequency table (table 22).

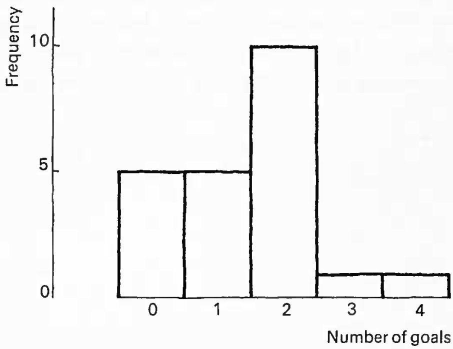


Figure 64

Ask what chance there would be of picking, at random with a pin, a team which had failed to score. Of the 22 teams, 5 failed to score; the chance is said to be 5 in 22 or 5/22.

This, the relative frequency, may be called the *probability estimate*.

$$\text{probability estimate} = \frac{\text{frequency of the event}}{\text{total number of events}}$$

So far, chance has been concerned with events in the past. What is the chance of picking a team which will fail to score in the *next* set of matches? Can the probability estimated on past results be used to forecast what might happen?

Other sequences of data of the same kind as the data used above should now be obtained and analysed. The data for First Division football matches for 4 more weeks are presented in table 23 and figure 65. For each week a histogram has been drawn. In this case, the same total number of events occur in each set of data. If the total number in each set changes, relative frequencies should be considered.

Number of teams scoring	**0 goal**	**1 goal**	**2 goals**	**3 goals**	**4 goals**	**5 goals**
Week A	4	8	4	5	0	1
Week B	7	6	6	1	1	1
Week C	6	7	6	3	0	0
Week D	4	8	8	1	1	0

Table 23
First Division results over four weeks.

Results considered week by week show only a little consistency. If, however, the results are compared month by month then, though the actual fluctuations are larger, the fraction of the teams failing to score, or scoring once, etc., is more steady. Again, it is the relative frequency that is being considered. For purposes of comparison, the frequencies for 3 periods of 4 weeks are given in table 24, together with the frequencies for a complete season. The relative frequencies are given in brackets.

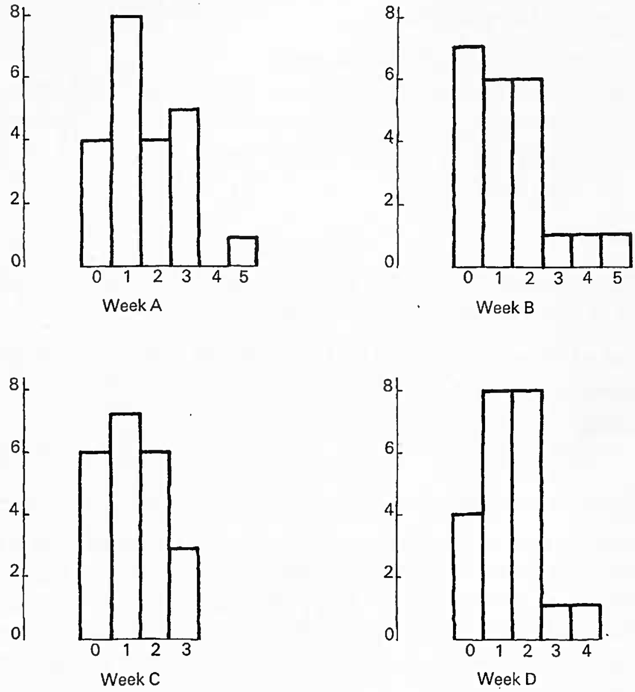


Figure 65

	Month A	Month B	Month C	Season
Number of teams failing to score	18 (0.31)	27 (0.345)	21 (0.24)	252 (0.273)
Number of teams scoring 1 goal	22 (0.38)	22 (0.28)	29 (0.33)	335 (0.362)
Number of teams scoring 2 goals	5 (0.085)	20 (0.255)	24 (0.27)	196 (0.212)
Number of teams scoring 3 goals	8 (0.14)	7 (0.09)	10 (0.11)	97 (0.105)
Number of teams scoring 4 goals	3 (0.05)	1 (0.015)	2 (0.025)	30 (0.032)
Number of teams scoring 5 goals	2 (0.035)	1 (0.015)	2 (0.025)	11 (0.012)
Number of teams scoring 6 goals	0	0	0	2 (0.002)
Number of teams scoring 7 goals	0	0	0	1 (0.001)
Total number of teams	58	78	88	924

Table 24
First Division results over three months and over the season.

Examples for section 9.1

Plot histograms to display the following data:

a For the lengths of sentences in a chapter of a book, indicated in table 25.

Number of words	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	41–45	46–50
Frequency	6	18	9	15	9	12	2	3	3	2

Table 25

b For a weak radioactive source, the counts indicated in table 26, recorded by a GM tube and scaler over 30-second intervals.

Range of count	100–109	110–119	120–129	130–139	140–149	150–159
Frequency	2	7	21	17	6	3

Table 26

9.2 Measuring probability

The work of section 9.1 should show that random fluctuations have a lessening effect on the relative frequency or probability estimate as the total number of events considered increases. In this section, the probability of a coin coming down head uppermost when tossed is measured. Students who object because 'the result is obvious' could toss a drawing-pin onto a hard surface and measure the probability of it ending point uppermost.

Groups of students could toss coins and record the sequence of heads and tails obtained over 100 throws. Plotting the excess of heads or tails at each stage in the sequence will result in charts showing considerable fluctuations (figure 66). The charts convey little except that the fluctuations do *not* decrease as the number of throws increases. Amongst the fluctuations, there will, no doubt, be some long runs of consecutive heads or tails, e.g. in the case shown in figure 66, there is a run of 6 consecutive heads and one of 7 tails. It is not possible at any stage to predict after a throw what the next will be.

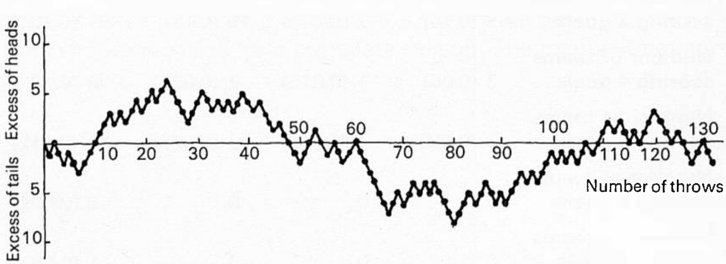


Figure 66

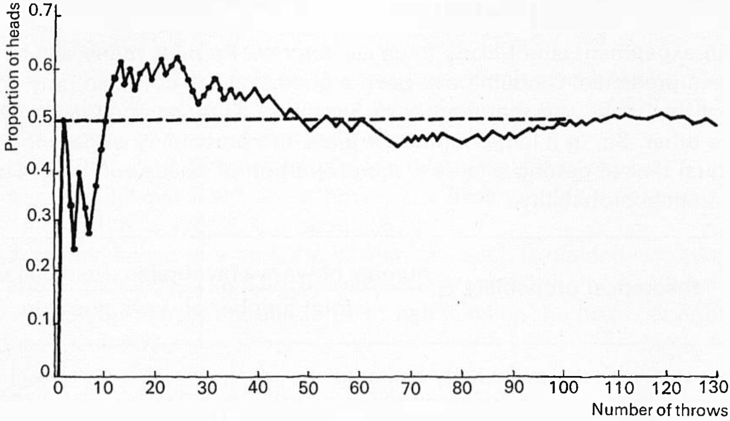


Figure 67

If, however, the relative frequency (number of heads/number of throws) is plotted (figure 67), the graph shows clearly that the fluctuations have a decreasing effect as the number of throws increases and that the proportion of heads obtained settles down to a number close to 0.5, i.e. in a large number of throws, very close to half of them will be heads. That proportion becomes more constant as the number of throws increases.

The results in table 27 measure the probability estimate of the coin used falling head uppermost when tossed.

Number of throws	Number of heads	Proportion
1 000	502	0.502
2 000	1013	0.507
3 000	1510	0.503
4 000	2029	0.507
5 000	2533	0.507
6 000	3009	0.502
7 000	3517	0.502
8 000	4035	0.504
9 000	4539	0.504
10 000	5068	0.507

Table 27
From Kerrich, J. E. (1946). An experimental introduction to the theory of probability. *Munksgaard, Copenhagen.*

9.3 Theoretical probability

The coin experiment is not likely to cause surprise. Perhaps many will think it could have been predicted. It could have been argued that the coin can only come to rest in one of two ways and that there is no reason to suppose one way is more likely than the other. So, in a large number of trials, the probability of getting a head must equal that of getting a tail, i.e. the proportion of heads will be $\frac{1}{2}$. This is called the theoretical probability.

$$\text{theoretical probability} = \frac{\text{number of ways a favourable event can occur}}{\text{total number of ways possible}}$$

Compare this with the probability estimate:

$$\text{probability estimate} = \frac{\text{frequency of favourable events}}{\text{total number of events}}$$

Intuitively, the theoretical probability will be regarded as the value the probability estimate will reach after a large number of events. It should be noted, however, that the argument depends on the ways being equally likely. For this to hold, the coin must be perfect and how can this be known unless it is tested by the method of section 9.2? Students who expect a result of 0.5 are assuming an ideal coin. In the drawing-pin experiment, the ways in which the pin can come to rest are not equally likely. That case does not involve a symmetry.

Common usage of the word 'probability' is often in the context of 'degree of belief' that an event will occur, e.g. a horse winning a race or rain falling tomorrow. The remark: 'There is a 50 per cent chance that it will rain tomorrow' is not usually an assessment of probability by the number of ways events can occur or have occurred in the past. It is usually nothing more than the expression of an opinion.

9.4 The probability scale

When a coin is tossed, it could come to rest head up or tail up or it might stand on its edge. In the test of section 9.2, standing on edge never happened (even in 10 000 throws) and so its probability is taken to be zero. Again, it is not possible for a two-headed penny to turn up tails and the probability of a tail is zero. Impossible events have a probability of 0.

With a two-headed penny, heads will turn up every time giving a probability of 1 – a certainty.

The scale of probability extends from impossibility to certainty, from 0 to 1.

The class should now be given practice in deducing theoretical probabilities for ideal cases.

Examples for sections 9.3 and 9.4

1 Consider a six-sided die.
a What is the total number of possible ways the die can land when thrown?
b What proportion of the total number of ways will give a 5 uppermost?
c What is the probability of a 5 being thrown?
d If the die were thrown 6 times, how many fives would you expect?
e If the die were thrown 1000 times, how many fives would you expect?

[**a** 6; **b** 1/6; **c** 1/6; **d** ?; **e** about 167.]

2 A ball is placed in a box, the bottom of which is divided into two equal areas, X and Y, by a line. The ball can move freely within the box. What fraction of the time would you expect the ball to be in half X when the box is continuously and randomly shaken?

[$\frac{1}{2}$.]

3 **a** At a fairground booth a 2p coin can be rolled down a small ramp onto a table on which are drawn a series of parallel lines 32 mm apart. If the coin comes to rest without touching or lying across a line, the player gets his coin back with a bonus of 3 coins. How much do you expect to win or lose if you try the game 2000 times? The diameter of a 2p coin should be taken to be 24 mm.

[Nothing.]

b If another set of parallel lines cross the first set at right angles, also at a separation of 32 mm, what would you expect then?

[Probability of winning = 1/16.
∴ In 2000 goes, you would expect to win 125 times, lose 1875 times.]

9.5 Combining probabilities

This section is concerned with the probability of two or more independent events occurring.

Consider a coin being tossed twice. What is the probability of getting two heads? These two events are independent of one another; the way the coin falls the first time has no effect on how it will fall on the second occasion. The possible ways the coin can fall are (H = head, T = tail):

HH : HT : TH : TT.

There are 4 possible ways, only one of which gives two heads. The probability of getting two heads is $\frac{1}{4}$.

The above problem is similar to finding the probability of two coins falling heads when both are tossed. There are 4 possible results and the probability of getting two heads is $\frac{1}{4}$. If the point needs making, diagrams such as those shown in figure 68 could be drawn.

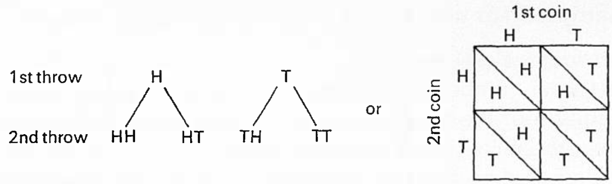


Figure 68

A point which should emerge from these problems is that, with independent events, the number of ways (and the probabilities) multiply.

The probability of getting a head and a tail is worth discussing. If the order matters, then there is only one way of being 'successful' and so the probability is $\frac{1}{4}$ (i.e. $\frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$). However, if the order does not matter, then there are two 'successful' results and the probability becomes $\frac{1}{2}$ (i.e. $\frac{1}{4}+\frac{1}{4} = \frac{1}{2}$). If the probability of event A happening is P_A and that of B happening is P_B, then the probability of A *and then* B happening is $P_A \times P_B$, whereas the probability of *either* A *or* B happening is $P_A + P_B$.

The problems for the previous section about tossing dice and about placing counters in a box merit further discussion. The former appears in Unit 5, *Atomic structure*, and the latter in Unit 9, *Change and chance*.

In Unit 5, there is an experiment in which 100 dice (or wooden cubes) are rolled and those showing a 6 (or a marked face) are removed and counted. The remainder are reshaken and again the 6s are removed and counted, and so on for about 10 throws. The experiment gives evidence of an exponential decay in the number of dice remaining after each throw. In the ideal case of very many trials, it would be expected that $\frac{1}{6}$ of those shaken would show up 6, so that the numbers remaining would decrease as below:

100 83 69 58 48 40 33 28 23 19 16.

The method by which these numbers are obtained indicate that the pattern is exponential.

Table 28 expresses the results in more general terms:

	Number before throwing	**Number showing 6**	**Number at end of throw**
1st throw	N	$\frac{1}{6}N$	$N-\frac{1}{6}N = \frac{5}{6}N$
2nd throw	$\frac{5}{6}N$	$\frac{1}{6}\,\frac{5}{6}N$	$\frac{5}{6}N-\frac{1}{6}\,\frac{5}{6}N = (\frac{5}{6})^2N$
3rd throw	$(\frac{5}{6})^2N$	$\frac{1}{6}(\frac{5}{6})^2N$	$(\frac{5}{6})^2N-\frac{1}{6}(\frac{5}{6})^2N = (\frac{5}{6})^3N$
⋮	⋮	⋮	⋮
nth throw	$(\frac{5}{6})^{n-1}N$	$\frac{1}{6}(\frac{5}{6})^{n-1}N$	$(\frac{5}{6})^nN$

Table 28

In terms of probabilities, the probability of throwing a 6 on the 3rd throw, but not before, is $\frac{1}{6}(\frac{5}{6})^2$, i.e. the product of the probabilities of not throwing a 6 on the first throw, not throwing a 6 on the second throw, and throwing a 6 on the third throw ($\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$). Similarly, the chance of not throwing a 6 at all in three throws is $(\frac{5}{6})^3$.

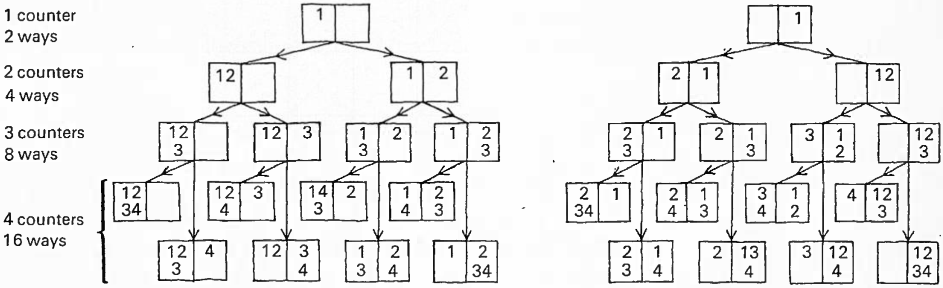


Figure 69

In the case of the 'counters-in-a-box' question, a tree diagram (figure 69) will help to clarify matters and show that the probabilities of independent events multiply.

There is an experiment in Unit 9, *Change and chance*, to test these predictions. The class place 4 counters at random in the halves of a box drawn on paper. The easiest way to do this is to spin a coin to decide which half each counter should go in, or to use a die and place the counter according to whether the die shows an odd or an even number. With a class of 16, there should be an average of 1 student each time getting all counters in one specified half. About 10 trials should be sufficient. It should be clear that the probability of any counter arrangement is 1/16, i.e. that the probabilities multiply ($1/2^4 = 1/16$).

Examples for section 9.5

1 A letter is selected at random from the word 'OPERATIONS'. What is the chance that it will be **a** a vowel **b** letter O?

[**a** 0.5; **b** 0.2.]

2 If a coin is tossed 3 times, what is the chance of getting **a** HHH **b** HTH **c** THT **d** two heads and a tail in any order?

[**a** 1/8; **b** 1/8; **c** 1/8; **d** 3/8.]

3 a What is the chance of throwing two sixes with two dice?

[1/36.]

b If a die is thrown twice what is the chance of successive sixes?

[1/36.]

c What is the chance of throwing less than 3 with a die?

[1/3.]

d What is the chance of throwing a total of 3 or less with two dice?

[1/12.]

e Construct a table (figure 70) showing all the possible results of throwing two dice. What is the most likely score and what is its probability?

[7; 1/6.]

	1	2	3	4	5	6
1	1 \ 1	1 \ 2	1 \ 3	1 \ 4	1 \ 5	1 \ 6
2	2 \ 1	2 \ 2	2 \ 3	2 \ 4	2 \ 5	2 \ 6
3	3 \ 1	3 \ 2	3 \ 3	3 \ 4	3 \ 5	3 \ 6
4	4 \ 1	4 \ 2	4 \ 3	4 \ 4	4 \ 5	4 \ 6
5	5 \ 1	5 \ 2	5 \ 3	5 \ 4	5 \ 5	5 \ 6
6	6 \ 1	6 \ 2	6 \ 3	6 \ 4	6 \ 5	6 \ 6

Figure 70

4 A box is divided into 2 halves, A and B (figure 71).

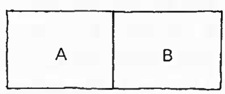


Figure 71

a If a counter may be placed in either half, how many ways are there of placing it in the box?

[2.]

b How many ways are there of placing 2 counters in the box?

[$4 = 2^2$.]

c How many ways are there with 3 counters?

[$8 = 2^3$.]

d How many ways are there with 4 counters?

[$16 = 2^4$.]

e What proportion of the ways of arranging the 4 counters in the box gives all counters in part A?

[1/16.]

f If the counters move about randomly within the box, how often would one expect to see all the four counters in half A?

[1/16 of the time.]

g What is the probability of all 4 counters being in half A?

[1/16.]

h What is the probability of 3 in half A and one in half B?

[1/4.]

i What is the probability of 2 in each half?

[3/8.]

5 A die is thrown several times.

a What is the probability of throwing a 6 with the first throw?

[1/6.]

b What is the probability of not throwing 6 with the first throw?

[5/6.]

c What is the probability of first throwing a 6 with the second throw, i.e. the probability of not throwing 6 with the first throw and then throwing 6 with the second throw?

[5/36.]

d What is the probability of throwing a 6 with the third throw but not before?

[25/216.]

e What is the probability of not throwing a 6 in three throws?

[125/216.]

f By considering the probabilities of first throwing a 6 on the first, second, and third throws, what is the probability of throwing at least one 6 in three throws?

[(1/6+5/36+25/216) = 91/216.]

g By considering the probability of *not* throwing a 6 in three throws, what is the probability of throwing at least one 6 in three throws?

[1−(125/216) = 91/216.]

6a A pack of cards (without jokers) is cut. What is the probability of cutting *1* a red suit *2* a diamond *3* a picture card *4* an ace *5* the ace of spades?

[*1* 1/2; *2* 1/4; *3* 3/13; *4* 1/13; *5* 1/52.]

b The pack is now cut on 4 occasions. What is the probability of cutting a picture card or ace on each occasion? What is the probability of not cutting a picture card or ace at all? What is the probability of cutting a picture card or ace at least on one occasion?

[$4/13)^4 = 0.009$; $(9/13)^4 = 0.23$; $(1-0.23) = 0.77$.]

7 A set of dominoes are shaken in a bag and one is drawn from the bag at random. What is the probability that at least one half of that domino will be blank?

[7/28 = 1/4.]

A second domino is picked without the first one being replaced. Find the probabilities of the following results.

a Both the first and the second were 'blanks'.

[$7/28 \times 6/27 = 1/18$.]

b The first, but not the second was a 'blank'.

[$7/28 \times 21/27 = 7/36$.]

c The second, but not the first was a 'blank'.

[$21/28 \times 7/27 = 7/36$.]

d Neither had a 'blank' on it.

[$21/28 \times 20/27 = 5/9$.]

9.6 Randomness – the most likely distribution

What is meant by 'random events', 'moving randomly'? In a random state of affairs, any one way (or arrangement) in which an event can occur is just as likely to occur as any other way. There are no restrictions; the choice is impartial. Rolling a perfect die could give random numbers of pips uppermost; a loaded die could not, for the die would tend to come to rest with its centre of gravity as low as possible and that way of settling would be more likely than the others. The probability of the face opposite to the loaded side ending uppermost would be greater than the probability of any one of the other faces finishing uppermost.

This does not mean that in randomness all *distributions* are equally probable. Consider the 4 counters placed in the divided box. There are 16 different ways of arranging the numbered counters between the halves A and B. Only one of these has all 4 counters in half A – the probability of that distribution is 1/16. The distribution of two counters in each half can occur in 6 different ways, giving a probability of 6/16.

The class might tabulate the different distributions and the probabilities as in table 29.

Arrangement of counters		**Probability**
Number in A	**Number in B**	
0	4	1/16
1	3	4/16
2	2	6/16
3	1	4/16
4	0	1/16

Table 29

The most likely distribution is 2 counters in each half, because this distribution can occur in the greatest number of ways (6). The least likely distribution is all the counters in one specified half – because there is only one way in which that can occur. The probability of finding 'all in one half' becomes increasingly smaller as the number of 'counters' increases, as the examples for the section show.

To reinforce the point, the data for 8 molecules in the box can be quoted. See table 30.

Arrangement of molecules		**Number of ways in 2^8 ways**	**Probability**
Number in A	**Number in B**	**in which this can occur**	
0	8	1	1/256
1	7	8	8/256
2	6	28	28/256
3	5	56	56/256
4	4	70	70/256
5	3	56	56/256
6	2	28	28/256
7	1	8	8/256
8	0	1	1/256

Table 30

Again the most likely distribution is the one in which the molecules are equally divided, and those distributions which are nearly equal to it have high probability too. The chance of getting all in one specified half is just one way in the total number of possible ways.

The following passage by L. Boltzmann could be used as a summary of these points:

> 'From an urn, in which many black and an equal number of white but otherwise identical spheres are placed, let 20 purely random drawings be made. The case that only black balls are drawn is not a hair less probable than the case that on the first draw one gets a black sphere, on the second a white, on the third a black, etc. The fact that one is more likely to get 10 black spheres and 10 white spheres in 20 drawings than one is to get 20 black spheres is due to the fact that the former event can come about in many more ways than the latter.'

From Boltzmann, L., translated by Brush, S. &. (1964). Lectures on gas theory. *Originally published by the University of California Press; reprinted by permission of The Regents of the University of California.*

Examples for section 9.6

1a Suppose there are 100 molecules of gas in a container. Over a long period, what fraction of that time will all the molecules spend, by chance, in one half?

[The number of ways of arranging 100 molecules between the two halves is 2^{100}, so the probability of 'all in one half' (i.e. a specified half) is 1 in 2^{100}. The value of 2^{100} can be found by taking logarithms.
$\lg 2^{100} = 100 \times 0.30 = 30$ so that $2^{100} = 10^{30}$ approximately.
If the molecules were observed every microsecond, then in 10^{30} μs (or 10^{24} s), you might expect to see the molecules all in one specified half on one occasion. But, if the Universe is perhaps 10^{10} years old (i.e. 3×10^{17} s), there hasn't been much of a chance yet!]

b At atmospheric pressure, the container might have in it 10^{22} molecules. What fraction of the time over a long period will all the molecules spend, by chance, in one specified half?

[A fraction 1 in $2^{10^{22}}$.]

[The number expressing this result has about 3 000 000 000 000 000 000 000 digits in it.]

2 Two dice are thrown. In how many ways can the total of the numbers uppermost be a prime number?

[2 in 1 way, 3 in 2 ways, 5 in 4 ways, 7 in 6 ways, and 11 in 2 ways – a total of 15 ways.]

What is the probability of throwing a prime number in this way?

[Probability = 15/36.]

3 In pile A, there are ten cards, of identical size, numbered 1 to 10, the numbers being hidden. In pile B, there are 5 similar cards numbered 1 to 5, the numbers also being hidden. One card is to be drawn from each pile.
What is the probability of drawing 3 and 7?
[1/50.]
What is the probability of getting a 5 and a 2?
[2/50.]
In how many ways can a total of 10 or more be obtained and what is the probability?
[20; 20/50.]
What is the probability of finding the total is less than 10?
[30/50.]

9.7 Binomial coefficients

Students who have done certain mathematics courses at O-level may recognize the pattern for the number of ways for the distributions in section 9.5 as binomial coefficients. Others may wonder how to obtain these figures without having to draw lengthy tree diagrams. It is not suggested that the binomial theorem be studied, but teachers might like to consider a small digression if students seem interested.

Suppose a distribution diagram (figure 72) is drawn for the tossing of a coin or for the placing of counters in a divided box. Letter A stands for a head uppermost or a counter placed in part A of the box, B for a tail uppermost or a counter in part B, the order in which the letters appear having no significance. For example AB means a head and a tail in any order. For AA and BB, only one way is possible, but for AB there are two ways (head followed by tail, or tail followed by head).

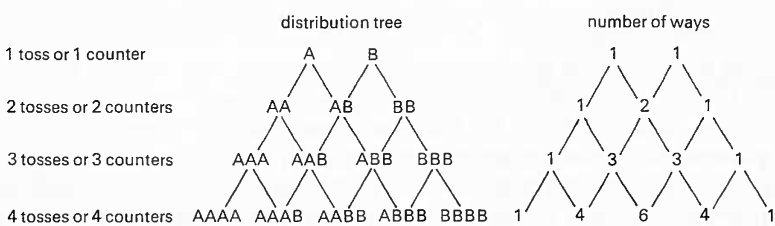


Figure 72

Now consider the distribution AAB. It is obtained either by first having AA and then B, or by having AB and then A. There is only one way of obtaining AA, and two ways for AB, so that there are 3 possible ways of getting AAB [A–A–B, A–B–A, B–A–A]. A 'number of ways' diagram can now be constructed, in which each number is the sum of the two numbers immediately above it. This is Pascal's Triangle (figure 73). It can be used to obtain the number of ways in which a given distribution can be achieved – provided the number of counters or tosses is not too large. The distribution diagram is not really needed because the distribution can also be deduced from Pascal's Triangle.

1 1	$2^1 = 2$
1 2 1	$2^2 = 4$
1 3 3 1	$2^3 = 8$
1 4 6 4 1	$2^4 = 16$
1 5 10 10 5 1	$2^5 = 32$
1 6 15 20 15 6 1	$2^6 = 64$
1 7 21 35 35 21 7 1	$2^7 = 128$
1 8 28 56 70 56 28 8 1	$2^8 = 256$
1 9 36 84 126 126 84 36 9 1	$2^9 = 512$
1 10 45 120 210 252 210 120 45 10 1	$2^{10} = 1\,024$
1 11 55 165 330 462 462 330 165 55 11 1	$2^{11} = 2\,048$
1 12 66 220 495 792 924 792 495 220 66 12 1	$2^{12} = 4\,096$
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1	$2^{13} = 8\,192$
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1	$2^{14} = 16\,384$
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1	$2^{15} = 32\,768$

Figure 73

Histograms should be drawn for a few of the distributions. It is possible to extend this work to the Normal distribution curve and the analysis of errors. A suitable development along those lines appears in the School Mathematics Project's book *Additional mathematics*, Part 2, Chapter 17. In the *Teachers' guide* for Unit 5, *Atomic structure*, experiment 5.12 is concerned with randomness and the uncertainty this introduces into a measurement of a count-rate. Details also appear in the *Students' laboratory book*. The class may welcome the opportunity to look at those ideas again.

Examples for section 9.7

1 Twelve counters are scattered randomly in a box divided into two halves. What is the probability of finding that the number in either half does not exceed the number in the other half by more than 4?

[Possible distributions are 4 and 8, 5 and 7, 6 and 6, 7 and 5, 8 and 4.
Number of ways $(495+792)\times 2+924 = 3498$
Total number of ways $= 2^{12} = 4096$.
Probability $= 3498/4096 \approx 35/41 = 0.85$.]

2 Draw a histogram to represent the probability of obtaining 0, 1, 2, 3 heads when 3 coins are tossed.
Do this for 4 coins (up to 4 heads), 5 coins (up to 5 heads), and 9 coins (9 heads).
In each of these cases, choose your scales so that the same area represents unit probability. For the case of 9 coins, use smaller column widths.
What would a histogram for a large number of coins look like?

9.8 Probability 'games'

Probability calculations and 'games' are sometimes used in attempts to account for the bulk properties of matter in terms of the behaviour of molecules. A 'game' of this kind was in fact used in the Nuffield O-level course when considering the diffusion of bromine molecules in air (Nuffield O-level Physics *Teachers' guide IV*, page 232).

The *Scientific American* Offprint 'Molecular motions' by Alder and Wainwright, provides useful reading at this point. One 'game' which is described is to find the distribution of distance between molecules by using a random number table to determine the position of molecules in an enclosure, and then measuring the separation of each pair of molecules. This is repeated many times to obtain the required distribution of distances. A similar game – for a two-dimensional enclosure – would be easy for the class to play. Details are given below.

Some insight into the realms of chance in science can be obtained if students read chapter 9 in *The laws of physics* by Rothman from which the following quotation has been taken.

> 'When physicists get down to measuring and describing the behaviour of particles, they find that they are dealing not with certainties but with probabilities. They cannot predict the behaviour of any single particle but only the *average* behaviour of a group of particles.
>
> 'The concept of probability has become a foundation-stone of modern physics.'

Exercise for section 9.8

a Place 4 'molecules' in the enclosure shown in figure 74 at positions determined by random number tables or by drawing cards out of a bag. If 7,0 is drawn, the 'molecule' is placed at the centre of the first square in column 7, etc.

b Measure the distance between each pair of 'molecules'.

c Assign a new place to each molecule in turn (by using the random number table or by drawing cards) and measure the distances again. If a 'molecule' lands on top of another, the move is not allowed and that molecule is considered not to have moved.

d Continue for a number of moves (at least 20) and draw a suitable histogram to show the distribution.

Of course, it makes it easier to collect the results quickly if several students play simultaneously and pool their readings at the end.

	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Figure 74

Section 10

Dynamics

There is no Unit in the Nuffield Advanced Physics course which is solely concerned with dynamics. The topics required are either treated in detail as part of the course when the need arises, or briefly revised if adequately covered in the Nuffield O-level course.

This book has given some attention to the revision of motion under constant acceleration, but there will probably be a need to give students additional practice with numerical problems covering a much wider range than that. Indeed, there may be a need for a more detailed second look at some of the O-level ideas and teachers should refer to the texts (Nuffield O-level Physics *Teachers' guides III, IV* and *V*) where appropriate. The topics which should be considered are:

1 Scalars and vectors.
2 Newton's Laws of motion, momentum, energy, and work.
3 Motion in a circle.
4 Simple harmonic motion.

Brief notes under these headings follow.

Elementary questions can be found in the questions books for the O-level course, particularly *Questions books IV* and *V*, and there are some rather harder examples in the Nuffield Advanced Physics *Students' books* for Units 2, 3, and 4. Rogers, *Physics for the inquiring mind* and PSSC *Physics* are good sources for suitable questions.

Scalars and vectors

Consideration of two displacements shows that numerical addition of the distances does not give the displacement from the starting point, and that geometrical addition by the parallelogram law is necessary. Velocity (displacement per unit time) and acceleration (change of velocity per unit time) behave similarly. Forces should also be considered, together with their resolution into two components. Chapters 2 and 3 of Rogers, *Physics for the inquiring mind*, will be found useful as a guide to teaching.

Newton's Laws of motion, momentum, energy, and work

This topic receives an extensive treatment in the O-level course and only a brief reminder of $F = ma$ should be needed. Students should know that force is also the rate of change of momentum and that momentum is a vector. Newton's Third Law and the conservation of momentum are important.

Chapters 7 and 8 of Rogers, *Physics for the inquiring mind*, are useful here, and PSSC *Physics* has chapters which cover the ground well.

The topics of energy and its conservation and transformation might also need consolidation. See particularly Units 2 and 3 of the Nuffield Advanced Physics course.

Motion in a circle

The class may need to see again the derivation of v^2/r for the inward acceleration. The vector method (Nuffield O-level Physics *Teachers' guide V*, page 47) should be used. This is also in Chapter 21 of Rogers, *Physics for the inquiring mind*. See Nuffield Advanced Physics, Unit 7.

Simple harmonic motion

Students may wish to go through some of the work of Unit 4 again and to widen their experience by looking at other examples of simple harmonic motion.

Appendix A
Books and other reading

Page numbers of references in this *Guide* appear in bold type.

Alder, B. J., and Wainwright, T. E. (1959) 'Molecular motions'. *Scientific American* Offprint no. 265. **114.**
Useful in section 9.8, 'Probability games'.

Austwick, K. (1962) *Logarithms.* Pergamon. **8.**

Gartside, S., and Kaye, D. (Ed. Roseveare, D.) (1969) *Square two,* volume 1. BBC Publications.
Quadling, D., Shuard, H., and Lawrance, A. (Ed. Roseveare, D.) (1970) *Square two,* volume 2. BBC Publications.
These books deal with most of the topics in this *Guide.*

Huff, D. (1965) *How to take a chance: the laws of probability.* Penguin. **98.**
A readable book for students.

Midlands Mathematical Experiment (1970) O-level *Book 2.* Harrap.
Provides useful questions for Sections 1, 2, 3, and 4.

PSSC (1965) *Physics.* 2nd edition. Heath **11, 16.**
Useful for Section 10, 'Dynamics'.

Rogers, E. M. (1960) *Physics for the inquiring mind.* Oxford University Press. **116, 117.**
Useful for Section 10, 'Dynamics'.

Rothman, M. A. (1966) *The laws of physics.* Penguin. **114.**
Useful for section 9.8, 'Probability games'.

School Mathematics Project (1968) *Additional mathematics book,* part 2. Cambridge University Press. **113.**
This book deals with most of the topics in this *Guide.*

School Mathematics Project (1964) *Book T.* Cambridge University Press.
School Mathematics Project (1965) *Book T4.* Cambridge University Press.
School Mathematics Project (1968) Supplement to *Books T* and *T4.* Cambridge University Press.
These provide useful questions for Sections 1, 2, 3, and 4.

Weaver, W. (1964) Science Study Series no. 24 *Lady Luck.* Heinemann. **98.**
A study of the theory of probability.

Appendix B
A syllabus for supplementary mathematics

Items marked * receive explicit attention within the Nuffield Advanced Physics course.

Indices and their manipulation – the power-of-ten notation.
Logarithms to base 10 and their use. The slide rule.
Ratio and scaling. The sine, cosine, and tangent of angle as a ratio.

Direct and inverse proportion graphically. The algebraic equations $y = kx$ and $y = k/x$. Dependence on more than one variable.
Linear graphs, gradient and intercept, $y = mx+c$.
Non-linear graphs, $y = kx^2$, $y = k/x^2$, $y = k/x$. Linear plots of these functions. The area 'under' a graph.
*Growth and decay functions.

*Small changes of variable and the gradient of a graph.
dy/dx as $\lim_{\Delta x \to 0} \Delta y/\Delta x$. Differentiation of x^n with a minimal treatment of $(1+x)^n$ when $x \ll 1$.
*Rates of change; turning points; d^2y/dx^2.
Small changes and 'errors'.

Sine and cosine functions treated graphically.
Derivative of $\sin\theta$, $\cos\theta$; the circular measure of angles and the small angle approximations.
Derivatives of $A \sin k\theta$ and $A \cos k\theta$; angular rotation.

Integration as reverse differentiation.
*Numerical reverse differentiation to solve simple equations,
*e.g. $dy/dx = kx$, $dy/dt = 5y$, $d^2y/dt^2 = k$, $d^2y/dt^2 = -\omega^2 y$.
Area under a graph and the $\int_0^x y\,dx$ notation. Its equivalence to reverse differentiation.
*Growth and decay patterns; the exponential variation.
*The numerical solution of $dN/dt = \pm kN$.
*The number e; Napierian logarithms.

*Random events and frequency. Relative frequency and theoretical probability. The probability scale.
Combining probabilities.
*Randomness and the most likely distribution. The binomial coefficients (Pascal's Triangle).

*Revision of dynamics; scalars and vectors.
*Newton's Laws of motion, momentum, energy; motion in a circle; simple harmonic motion.

Index

Where the page reference is to an example, it is italicized.

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This Teachers' Guide, *Supplementary mathematics*, is intended to be used in conjunction with the other Teachers' Guides for the Nuffield Advanced Physics course. It provides resources for teachers who need to offer a course in mathematics parallel to Advanced Physics for students not taking an Advanced level mathematics course.

The first three sections are mainly revision of ideas and techniques from Ordinary level. Section 1 is on computation, including power of ten notation; section 2 concerns function and proportionality; section 3 is on linear graphs.

The remainder of the book concerns mathematical ideas and tools developed at least in part in the Nuffield Advanced Physics course. The material offered here is intended to supplement that in the Physics course. The sections are entitled 'Non-linear graphs', 'Differentiation', 'Sine and cosine graphs', 'Integration', 'Exponential variations', 'Chance', and 'Dynamics'.

Two Appendices list other relevant books, and provide an outline of the supplementary mathematics course in the form of a syllabus.

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